

Impedance and admittance measurement using discrete scalar voltage samples

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Abstract

The resistive and reactive components of an impedance, including the sign of the reactance, can be determined by placing a resistance and a reactance in series with the impedance and making five scalar voltage measurements across pairs of points in the resulting network. It is also possible, using suitable combinations of the same 5 measurements, to determine impedance magnitude, phase tangent, conductance, susceptance, power factor, and (by making the reference resistance equal to the target load resistance) reflection coefficient and SWR. This article gives expressions for all of these quantities; and includes an error analysis in each instance to show how a standard deviation can be calculated. The error functions are well-behaved except for those related to the voltage reflection coefficient, the expression for which is not differentiable when its value is in the region of zero. A solution to that problem is to place a potential divider across the generator and make an additional bridge voltage measurement, this configuration being the same as is used in antenna analysers and return loss bridges.

Although not suitable for permanent insertion into a transmission line; the series network can be used for simultaneous determination of load resistance, conductance and phase tangent, as is required for guaranteed convergence of impedance matching procedures involving series and parallel reactances.

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Introduction

In an article in QST magazine in 1965, Heathkit design engineer Doyle Strandlund described a method by which the resistance and reactance (including sign) of an unknown impedance can be determined using only scalar measurements². The technique involves placing a resistance and a reactance in series with the impedance, driving the combination from the output of a radio transmitter (via an attenuator); and using shunt-diode detectors to measure the voltages across the series elements and the voltages from the junctions between the elements to ground. The readings from such detectors are, of course, peak voltage magnitudes (neglecting diode non-ideality), devoid of phase information; but the data obtained can nevertheless be analysed to find the real and imaginary components of the impedance under test.

Strandlund offered a graphical method for determining the test impedance; but the problem can easily be turned into a set of formulae, and so the chart-plotting approach fell out of favour with the advent of the personal computer³. The method can also be updated by the inclusion of diode non-linearity and temperature compensation circuits, as was outlined by M E Gruchalla⁴. Gruchalla also gave a derivation of formulae for R and X, but did not include an error analysis.

To see the circuit only as a type of impedance meter however misses many of its features. It might, more interestingly, be regarded as a prototype for set of circuit configurations that can be used to acquire data for the calculation of impedance ($R+jX$), admittance ($G+jB$), magnitude, phase angle and power factor. Moreover, if the reference resistance is chosen to be the same as the target load resistance, it can be used to calculate reflection coefficient and SWR. Incidentally, with the reference reactance set to zero, it corresponds to the method by which low-cost antenna analysers extract impedance magnitude from the reference and measurement arms of an unbalanced Wheatstone bridge.

Here we re-analyse the circuit for its stock of possibilities, derive the error functions, and show how it can be used to make a match-meter, i.e., a measuring device for R, G and $\tan\phi$ (i.e., X/R) that can be used to guarantee the convergence of impedance matching procedures involving reactive π , T and L-networks.⁵

2 **Amateur measurement of $R+jX$** , Doyle Strandlund, QST, June 1965, p24-27.

3 **Measuring RF Impedance Using the Three Meter Method and a Computer**, Peter Dodd, ARRL Antenna Compendium, Vol 4. 1995, p175-179.

More on the 3-Meter Impedance Measuring Bridge, Peter Dodd, ARRL Antennan Compendium, Vol 5, 1996, p144-147.

['Three meter method' is a misnomer. The articles are about the five voltage method under discussion here].

4 **Complex Impedance Measurement Using only Scalar Voltage Measurements**, M E Gruchalla, Communications Quarterly (CQ), Fall (Oct.) 1998, p33-43.

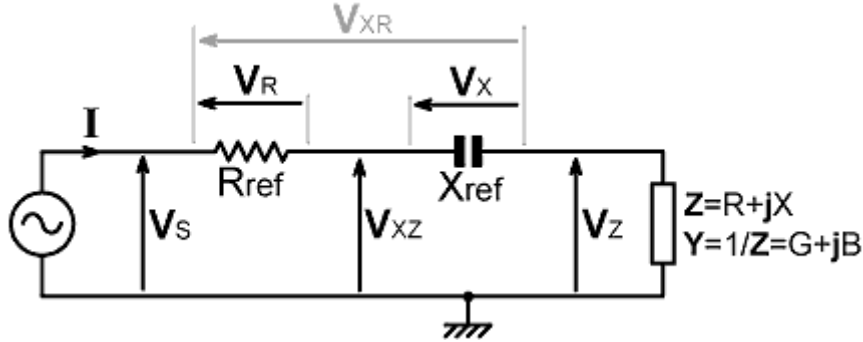
A comment on Complex Impedance Measurements, Peter Dodd, CQ, Spring 1999, p3.

5 **Automatic Tuning of Antennae**, M J Underhill and P A Lewis, SERT Journal, Vol 8, Sept. 1974, p183-184.

See also: [www.g3ynh_info/zdocs/z_matching/index.html](http://www.g3ynh.info/zdocs/z_matching/index.html)

1. Network voltages

The theoretical network is shown below, with the five voltages that are usually measured marked in black. The reference reactance X_{ref} is drawn as a capacitor because it should ideally be a pure reactance, in which case a low-loss capacitor seems a better choice than an inductor. An additional voltage V_{XR} is also shown, in grey. This (as was pointed out by Strandlund) might possibly be used to compute a correction for the loss resistance of the reactance (although it might be difficult to achieve a statistically valid result).



With suitable attention to the DC return arrangements, all of the voltages shown can be measured using shunt-diode detectors and a high-impedance DC voltmeter. Since the detectors cannot preserve phase information however, what we actually measure (neglecting scale factors, and assuming linearity correction) is a set of magnitudes: $|V_S|$, $|V_R|$, $|V_{XZ}|$, $|V_X|$, $|V_Z|$ and, optionally $|V_{XR}|$.

Now notice that all of the elements are in series and so share a common current I . Hence, for example, with the impedance under test defined as $R + jX$, we can write an expression for the source voltage V_S as:

$$V_S = I [R + R_{\text{ref}} + j(X + X_{\text{ref}})]$$

Also, noting that the magnitude of the product of two phasors is the product of the magnitudes, we can represent the detected voltage as follows:

$$|V_S| = |I| \sqrt{(R + R_{\text{ref}})^2 + (X + X_{\text{ref}})^2}$$

We can also convert this into an impedance magnitude analogue by noting that $|I|$ is just a common scale factor, so that:

$$\frac{|V_S|}{|I|} = \left| \frac{V_S}{I} \right| = \sqrt{(R + R_{\text{ref}})^2 + (X + X_{\text{ref}})^2}$$

Also, it will usually be convenient to eliminate square-root symbols and work with the squares of magnitudes, thus:

$$\left| \frac{V_S}{I} \right|^2 = (R + R_{\text{ref}})^2 + (X + X_{\text{ref}})^2$$

and sometimes we will want to multiply-out the brackets:

$$\left| \frac{V_S}{I} \right|^2 = R^2 + R_{\text{ref}}^2 + 2R R_{\text{ref}} + X^2 + X_{\text{ref}}^2 + 2X X_{\text{ref}}$$

Using this approach we can write-out a set of formulae for the various available analogues. They are as follows:

Table 1.

$\left \frac{\mathbf{V}_S}{\mathbf{I}} \right ^2 = (\mathbf{R} + \mathbf{R}_{\text{ref}})^2 + (\mathbf{X} + \mathbf{X}_{\text{ref}})^2 = \mathbf{R}^2 + \mathbf{R}_{\text{ref}}^2 + 2\mathbf{R}\mathbf{R}_{\text{ref}} + \mathbf{X}^2 + \mathbf{X}_{\text{ref}}^2 + 2\mathbf{X}\mathbf{X}_{\text{ref}}$	(1)
$\left \frac{\mathbf{V}_{\text{XZ}}}{\mathbf{I}} \right ^2 = \mathbf{R}^2 + (\mathbf{X} + \mathbf{X}_{\text{ref}})^2 = \mathbf{R}^2 + \mathbf{X}^2 + \mathbf{X}_{\text{ref}}^2 + 2\mathbf{X}\mathbf{X}_{\text{ref}}$	(2)
$\left \frac{\mathbf{V}_Z}{\mathbf{I}} \right ^2 = \mathbf{R}^2 + \mathbf{X}^2$	(3)
$\left \frac{\mathbf{V}_R}{\mathbf{I}} \right ^2 = \mathbf{R}_{\text{ref}}^2$	(4)
$\left \frac{\mathbf{V}_X}{\mathbf{I}} \right ^2 = \mathbf{X}_{\text{ref}}^2$	(5)
$\left \frac{\mathbf{V}_{\text{XR}}}{\mathbf{I}} \right ^2 = \mathbf{R}_{\text{ref}}^2 + \mathbf{X}_{\text{ref}}^2$	(6)

There are also two important combinations that will find recurrent use:

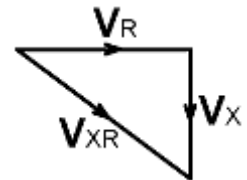
Table 2.

$\frac{ \mathbf{V}_S ^2 - \mathbf{V}_{\text{XZ}} ^2 - \mathbf{V}_R ^2}{ \mathbf{I} ^2} = 2\mathbf{R}\mathbf{R}_{\text{ref}}$	(1) - (2) - (4)
$\frac{ \mathbf{V}_{\text{XZ}} ^2 - \mathbf{V}_Z ^2 - \mathbf{V}_X ^2}{ \mathbf{I} ^2} = 2\mathbf{X}\mathbf{X}_{\text{ref}}$	(2) - (3) - (5)

By considering the expressions in the tables above, we can see that any formula for an impedance or admittance attribute or component must be formed as a quotient (a fraction) in order to eliminate the current (and the neglected scale factor due to rectification). Also, equations (4) and (5) are the current-squared analogues, because they can be traded for $|\mathbf{I}|^2$ against the known values of the reference components, for which reason they will be seen to appear in the denominators of the expressions for X and R.

Note that the phasors \mathbf{V}_R and \mathbf{V}_X should be at right angles to each other. Therefore \mathbf{V}_{XR} is the hypotenuse of the right-angled triangle they should form. This gives a checksum for the purity of the reference elements, i.e., by Pythagoras:

$$|\mathbf{V}_R|^2 + |\mathbf{V}_X|^2 = |\mathbf{V}_{\text{XR}}|^2$$



Any statistically significant breakdown of this relationship signifies a problem with the reference standards.

2. Resistance

By inspection of tables 1 & 2, the expression for R is obtained using [(1) - (2) - (4)] / (4) , which gives:

$$\frac{(|\mathbf{V}_S|^2 - |\mathbf{V}_{XZ}|^2 - |\mathbf{V}_R|^2) / |\mathbf{I}|^2}{|\mathbf{V}_R / \mathbf{I}|^2} = \frac{2R}{R_{\text{ref}}}$$

The common current cancels throughout and so we have:

$$R = \frac{R_{\text{ref}}}{2} \frac{(|\mathbf{V}_S|^2 - |\mathbf{V}_{XZ}|^2 - |\mathbf{V}_R|^2)}{|\mathbf{V}_R|^2} \quad (2.1)$$

which simplifies to:

$$R = \frac{R_{\text{ref}}}{2} \left(\frac{|\mathbf{V}_S|^2 - |\mathbf{V}_{XZ}|^2}{|\mathbf{V}_R|^2} - 1 \right) \quad (2.1a)$$

A special property of this formula is that it does not use the voltages $|\mathbf{V}_X|$ and $|\mathbf{V}_Z|$. In fact, if we were to set X_{ref} to zero (i.e., short it out), it would seem to make no difference to the result. This means that the resistive component of an impedance can be determined by making scalar voltage measurements on the the reference and measuring arms of an unbalanced Wheatstone bridge.

Although X_{ref} makes no difference to this calculation however, its value, as we will see in the next section, does have an effect on the accuracy of the result.

2a. Error analysis and simulation

If we are to make use of any of the quantities extracted from the voltage measurements, we must attach an uncertainty (i.e., usually, a standard deviation) to every result. Therefore it is necessary to perform an error analysis in each instance. A carefully explained analysis of the resistance formula will serve to demonstrate the technique, which will thereafter be applied without further justification. The error function, as in every other instance, will also guide us in how to optimise the measurement parameters to get the most accurate reading.

Whenever a quantity is determined by combining a number of measurements, each measurement makes a contribution to the uncertainty in the result. The uncertainty contribution from a given measurement is equal to the rate of change of the quantity with respect to the measurement, multiplied by the uncertainty in the measurement. Thus, looking at the resistance formula, equation (2.1), and using a fairly standard statistical notation, we can write:

$$\delta R_{R_{\text{ref}}} = \sigma_{R_{\text{ref}}} \frac{\partial R}{\partial R_{\text{ref}}}$$

Where $\delta R_{R_{\text{ref}}}$ is the uncertainty contribution to R due to uncertainty in R_{ref} , $\sigma_{R_{\text{ref}}}$ is the absolute (rather than the percentage) standard deviation of the known value of R_{ref} , and $\partial R / \partial R_{\text{ref}}$ is the partial derivative of R with respect to R_{ref} (i.e., the result of differentiating R with respect to R_{ref} with all of the other variables held constant). The expression is true provided that the formula is

differentiable in the region of interest, and provided that the parameter uncertainty is reasonably small compared to the magnitude of the derivative.

Now, repeating the process for the other parameters (keeping the expressions legible by dropping the magnitude brackets in subscripts) we get:

$$\delta R_{vs} = \sigma_{vs} \frac{\partial R}{\partial |V_s|}$$

$$\delta R_{vxz} = \sigma_{vxz} \frac{\partial R}{\partial |V_{xz}|}$$

and

$$\delta R_{vr} = \sigma_{vr} \frac{\partial R}{\partial |V_r|}$$

Thus, in the case of resistance, we have four uncertainty contributions, which must be incorporated into the formula for the standard deviation. How to do that depends on whether the measurements are correlated or uncorrelated. If two measurements are correlated, then the uncertainty in one is related to the uncertainty in the other. This can happen, for example, when the two measurements are made using the same meter. It then depends on the signs of the derivatives, whether these correlated errors augment or tend to cancel each other. If we take pains to remove such systematic errors however, then what is left is random uncertainty; and in that case, the error in one quantity is not influenced by the errors in any of the others. When all of the errors are unrelated, then they are orthogonal, i.e., their contributions act like a set of vectors in a multi-dimensional space, each at right angles to all of the others. Hence, we can find the magnitude of the overall uncertainty by successive application of Pythagoras' theorem. This operation is accomplished simply by adding the squares of all of the uncertainty contributions and taking the square root⁶. Hence, for the resistance measurement:

$$\sigma_R = \sqrt{\left(\sigma_{R_{ref}} \frac{\partial R}{\partial R_{ref}}\right)^2 + \left(\sigma_{vs} \frac{\partial R}{\partial |V_s|}\right)^2 + \left(\sigma_{vxz} \frac{\partial R}{\partial |V_{xz}|}\right)^2 + \left(\sigma_{vr} \frac{\partial R}{\partial |V_r|}\right)^2}$$

The derivatives can mostly be determined by inspection, and a check on the logic can be had by noting that the quantities in brackets in the expression above (prior to squaring) have the same dimensions as the overall uncertainty. An absolute measurement uncertainty has the dimensions of the measurement, and that dictates the dimensions of the derivative. A derivative that relates a quantity in volts to a quantity in ohms (for example) must have dimensions of [ohms/volt] (i.e., [volts] × [ohms/volt] = [ohms]).

If we let:

$$U_R = \frac{|V_s|^2 - |V_{xz}|^2}{|V_r|^2} - 1$$

then, using the standard differentiation formula: $\frac{d(a x^n)}{dx} = a n x^{n-1}$;

⁶ This is explained in more detail in the article **Scientific Data Analysis** (by DWK), http://www.g3ynh.info/zdocs/math/data_analy.pdf

$$\frac{\partial R}{\partial R_{\text{ref}}} = \frac{U_R}{2} \quad [\text{dimensionless}]$$

$$\frac{\partial R}{\partial |V_S|} = \frac{|V_S| R_{\text{ref}}}{|V_R|^2} \quad [\text{ohms / volt}]$$

$$\frac{\partial R}{\partial |V_{XZ}|} = -\frac{|V_{XZ}| R_{\text{ref}}}{|V_R|^2} \quad [\text{ohms / volt}]$$

$$\frac{\partial R}{\partial |V_R|} = -\frac{R_{\text{ref}} (|V_S|^2 - |V_{XZ}|^2)}{|V_R|^3} \quad [\text{ohms / volt}]$$

There is not much point in combining all of this into a single long expression, but we can collect it all for easy reference as follows:

$R = \frac{U_R R_{\text{ref}}}{2}$ <p>where:</p> $U_R = \frac{ V_S ^2 - V_{XZ} ^2}{ V_R ^2} - 1$	(2.1b)
$\sigma_R = \sqrt{\left(\sigma_{R_{\text{ref}}} \frac{\partial R}{\partial R_{\text{ref}}} \right)^2 + \left(\sigma_{V_S} \frac{\partial R}{\partial V_S } \right)^2 + \left(\sigma_{V_{XZ}} \frac{\partial R}{\partial V_{XZ} } \right)^2 + \left(\sigma_{V_R} \frac{\partial R}{\partial V_R } \right)^2}$ <p>where</p> $\frac{\partial R}{\partial R_{\text{ref}}} = \frac{U_R}{2}$ $\frac{\partial R}{\partial V_S } = \frac{ V_S R_{\text{ref}}}{ V_R ^2}$ $\frac{\partial R}{\partial V_{XZ} } = -\frac{ V_{XZ} R_{\text{ref}}}{ V_R ^2}$ $\frac{\partial R}{\partial V_R } = -\frac{R_{\text{ref}} (V_S ^2 - V_{XZ} ^2)}{ V_R ^3}$	(2.1u)

What the error function **(2.1u)** tells us by inspection is that the uncertainty of the measurement will explode when the voltage across the reference resistor is small relative to the other voltages. Apart from that however, the easiest way to find-out more is to simulate the measurement process over a

range of parameters using realistic standard deviations for the reference resistance and the voltage measurements. This has been done in the Open Document Spreadsheet⁷ file [YZ_scalar_sim.ods](#) (sheet 1).

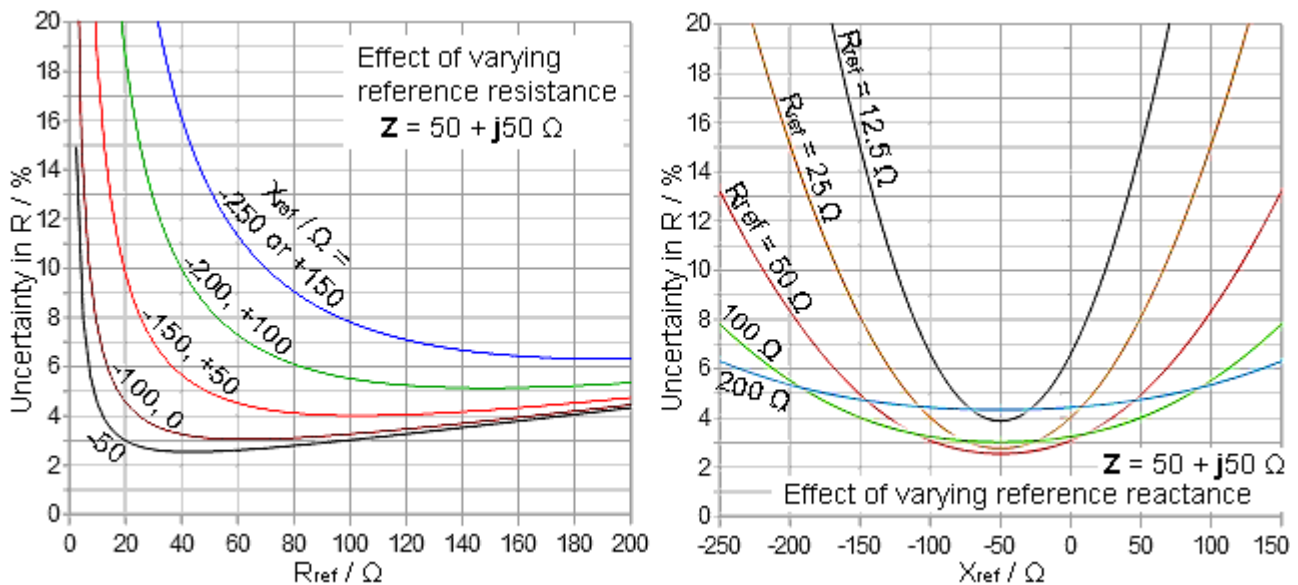
One reason for calculating intermediate results for the error function, rather than evaluating the whole thing in one go, is that there is a great deal to be learned by seeing the numerical values of the derivatives. Notice in the quantity U_R in formula (2.1b), that the process of calculating R involves subtracting two fairly large quantities to obtain a small quantity, and then inflating the value again by performing a division. The problem with such procedures is that the measurement errors are largely random, and so they are not suppressed by the subtraction process, but they are amplified by the division. Hence the derivatives of the resistance with respect to the measured voltages are large. U_R itself is fairly small however, and so the derivative of the resistance with respect to the reference resistance is small. The outcome is that the measurement is not strongly affected by the uncertainty in R_{ref} , which is a pity because this resistance is easily measured using a good DMM. Instead, it turns out that accurate voltage measurement is critical.

The spreadsheet is easily modified, but for the sake of this investigation, simulations were carried out in the region of $50\ \Omega$, with an assumed standard deviation (SD) of resistance measurement of 0.1%, and an assumed SD of voltage measurement of 0.5%. Using a straight percentage uncertainty is not necessarily realistic because, in addition to a scale error, most meters have an offset or zeroing error and ADCs have a quantisation error. This means that measurements become more inaccurate, in percent, for small readings. The consequence of using only a fixed percentage error is therefore that the result does not vary with source voltage, whereas in reality it will get worse at low voltages. A source voltage of 10 V was adopted however as a reasonable compromise; usually sufficient to overwhelm any offset errors, in observance of the need to avoid excessive power dissipation, and well within the V_{RM} limit of the commonly-used 1N5711 Schottky detector diode.

The simulation procedure involves using the source voltage and the input values of the unknown and reference resistances and reactances to calculate the magnitude of the common current. The current is then used to calculate the voltages that will appear across the measurement points. These voltages are then combined with the input percentage standard deviation to obtain the absolute SD of each voltage (i.e., the SD in volts)⁸. The voltages are also used to calculate the resistance R (as a check of the coding), the function U_R , and the derivatives. The derivatives are combined with the absolute voltage uncertainties in the error function σ_R , and the result is converted back into a percentage for graph plotting. The spreadsheet is configured so that either R , R_{ref} , X , or X_{ref} can be used as the x-axis, so that the effect of varying one parameter while holding the others constant can be seen. Constant reactance, of course, is not usually realistic, but it can be contrived by component adjustment or by changing frequency.

⁷ Opens in Open Office, available free from www.openoffice.org

⁸ This is the point at which offset and quantisation errors (which are fixed quantities for a given measurement method) might also be added.



The findings from the simulation are fairly well summarised by the graphs above. With voltage measurements accurate to $\pm 0.5\%$, the best possible uncertainty in the resistance determination is just a little over $\pm 2.5\%$ (SD). The circumstances under which this can be obtained however are somewhat restricted. The best outcome occurs when there is a conjugate match between the impedance under test and the reference elements. Thus, in the set of simulations shown in the graphs, where the test impedance is $50 + j50 \Omega$, the optimum reference resistance is 50Ω , and the optimum reference reactance is -50Ω . As can be seen in the right-hand graph, the degradation of accuracy can be particularly severe if R_{ref} is substantially less than R , but the error curve flattens out, and un-cancelled reactance is less deleterious, when R_{ref} is greater than R .

The simulation also confirms the qualitative observation made earlier, which is that it is best to make measurements in circumstances that give a relatively large voltage across R_{ref} . Thus the relatively flat error curve when R_{ref} is several times larger than R .

Overall, it is evident that the method is a little too idiosyncratic to be adopted as a general-purpose AC resistance measurement technique; but it can give reasonable accuracy under some circumstances, and it has the virtue that the accuracy will improve progressively if the purpose of measuring the test impedance is so that it can be adjusted to match a target value. In other words, the method is the basis of an antenna-system analyser, as those who have been using it for that purpose will obviously attest. An assertion made elsewhere however, that the value of the reference reactance is unimportant, is shown to be incorrect.

A good way to verify a spreadsheet analysis, incidentally, is to code the problem independently and check that both methods agree. Also, a large array of intermediate results is not needed in practice. Consequently, the following Open Office Basic⁹ macros were written (and they do indeed confirm the calculations discussed earlier). The calling arguments should be self-explanatory.

⁹ Open Office BASIC programming guide can be found at wiki.openoffice.org/wiki/Documentation/BASIC_Guide. To view, edit and copy the basic macros in the accompanying spreadsheet file, go to the 'Tools / Macros / Organise Macros / OpenOffice Basic' menu.. and select 'Edit'.

```

Function Scalarvolt_R(ByVal Rref as double, Vs as double, Vxz as double, _
VR as double) as double
'Calculates the resistive component (R) of an impedance using the scalar voltage method
Scalarvolt_R = 0
if VR = 0 then exit function
Scalarvolt_R = 0.5*Rref*(( (Vs^2 - Vxz^2) / VR^2 ) - 1)
End function

Function SV_sigmaR(ByVal sigmaRref as double, Rref as double, sigmaV as double, _
offsetV as double, Vs as double, Vxz as double, VR as double) as double
'calculates a standard deviation for R
'SigmaRref and sigmaV are input as percentages and converted into absolute uncertainties.
'OffsetV is the voltage measurement offset (or zeroing) error, in volts.
'The function returns an absolute uncertainty (in Ohms).
If VR = 0 then
SV_sigmaR = 1E12
Else
sigmaRref = Rref*sigmaRref/100
Dim sigmaVs as double, sigmaVxz as double, sigmaVR as double, UR as double
sigmaVs = Vs*sigmaV/100 + offsetV
sigmaVxz = Vxz*sigmaV/100 + offsetV
sigmaVR = VR*sigmaV/100 + offsetV
UR = ((Vs^2 - Vxz^2) / VR^2) - 1
Dim derivRref as double, derivVs as double, derivVxz as double, derivVR as double
derivRref = UR / 2
derivVs = Vs*Rref/VR^2
derivVxz = -Vxz*Rref/VR^2
derivVR = -Rref*((Vs^2 - Vxz^2)/VR^3)
SV_sigmaR = Sqr((sigmaRref*derivRref)^2 + (sigmaVs*derivVs)^2 _
+ (sigmaVxz*derivVxz)^2 + (sigmaVR*derivVR)^2)
End if
End function

```

Note that if $|V_R| = 0$, then the uncertainty in R is infinite, and so the error function returns a value of $10^{12} \Omega$ to ensure that continuous line graphs are plotted correctly. Note also that voltage offset error can be included if so desired. For example, for a 10-bit ADC with a 10 V reference and an offset error of ± 1 LSB, the offset error in volts is $10/1024 = 0.01$ V.

3. Reactance: 3-V method

There are two ways in which a reactance formula can be obtained. The most obvious choice is the one that produces an expression that is the same as the resistance formula except for a change of variables. From the analogues in [tables 1 & 2](#), the combination is [(2) - (3) - (5)] / (5) . This gives:

$$\frac{(|\mathbf{V}_{xz}|^2 - |\mathbf{V}_z|^2 - |\mathbf{V}_x|^2)}{|\mathbf{V}_x|^2} = \frac{2X}{X_{\text{ref}}}$$

Rearranging to bring out the reactance then gives:

$$X = \frac{X_{\text{ref}}}{2} \frac{(|\mathbf{V}_{xz}|^2 - |\mathbf{V}_z|^2 - |\mathbf{V}_x|^2)}{|\mathbf{V}_x|^2} \quad (3.1)$$

which can be simplified:

$$X = \frac{X_{\text{ref}}}{2} \left(\frac{|\mathbf{V}_{xz}|^2 - |\mathbf{V}_z|^2}{|\mathbf{V}_x|^2} - 1 \right) \quad \text{Three voltage (3-V) method} \quad (3.1a)$$

This uses three voltage measurements to determine X, and so we will refer to it as the 3-V method. Also, unlike most reactance measuring methods that rely on scalar voltage samples, this gives us the sign of the reactance provided that we know the sign of X_{ref} .

The error function, by the same logic as used in the previous section, is:

$$\sigma_X = \sqrt{\left(\sigma_{X_{\text{ref}}} \frac{\partial X}{\partial X_{\text{ref}}} \right)^2 + \left(\sigma_{V_{xz}} \frac{\partial X}{\partial |\mathbf{V}_{xz}|} \right)^2 + \left(\sigma_{V_z} \frac{\partial X}{\partial |\mathbf{V}_z|} \right)^2 + \left(\sigma_{V_x} \frac{\partial X}{\partial |\mathbf{V}_x|} \right)^2}$$

If we let:

$$U_X = \frac{|\mathbf{V}_{xz}|^2 - |\mathbf{V}_z|^2}{|\mathbf{V}_x|^2} - 1$$

Then the derivatives are identical in form to those for the resistance case:

$$\frac{\partial X}{\partial X_{\text{ref}}} = \frac{U_X}{2} \quad [\text{dimensionless}]$$

$$\frac{\partial X}{\partial |\mathbf{V}_{xz}|} = \frac{|\mathbf{V}_{xz}| X_{\text{ref}}}{|\mathbf{V}_x|^2} \quad [\text{ohms / volt}]$$

$$\frac{\partial X}{\partial |\mathbf{V}_z|} = -\frac{|\mathbf{V}_z| X_{\text{ref}}}{|\mathbf{V}_x|^2} \quad [\text{ohms / volt}]$$

$$\frac{\partial X}{\partial |\mathbf{V}_x|} = -\frac{X_{\text{ref}} (|\mathbf{V}_{xz}|^2 - |\mathbf{V}_z|^2)}{|\mathbf{V}_x|^3} \quad [\text{ohms / volt}]$$

Collecting it all we have:

$X = \frac{U_x X_{\text{ref}}}{2}$ <p>where:</p> $U_x = \frac{ \mathbf{V}_{xz} ^2 - \mathbf{V}_z ^2}{ \mathbf{V}_x ^2} - 1$	<p>(3.1b)</p> <p>3-V</p>
$\sigma_x = \sqrt{\left(\sigma_{X_{\text{ref}}} \frac{\partial X}{\partial X_{\text{ref}}}\right)^2 + \left(\sigma_{V_{xz}} \frac{\partial X}{\partial \mathbf{V}_{xz} }\right)^2 + \left(\sigma_{V_z} \frac{\partial X}{\partial \mathbf{V}_z }\right)^2 + \left(\sigma_{V_x} \frac{\partial X}{\partial \mathbf{V}_x }\right)^2}$ $\frac{\partial X}{\partial X_{\text{ref}}} = \frac{U_x}{2}$ $\frac{\partial X}{\partial \mathbf{V}_{xz} } = \frac{ \mathbf{V}_{xz} X_{\text{ref}}}{ \mathbf{V}_x ^2}$ $\frac{\partial X}{\partial \mathbf{V}_z } = -\frac{ \mathbf{V}_z X_{\text{ref}}}{ \mathbf{V}_x ^2}$ $\frac{\partial X}{\partial \mathbf{V}_x } = -\frac{X_{\text{ref}} (\mathbf{V}_{xz} ^2 - \mathbf{V}_z ^2)}{ \mathbf{V}_x ^3}$	<p>(3.1u)</p>

An error simulation for the 3-voltage method for reactance is given in the spreadsheet file [YZ_scalar_sim.ods](#) (sheet 2). Once again, the spreadsheet calculation is verified against separately coded Basic macro functions. In [section 5](#) however, we will also examine an alternative four voltage (4-V) method that eliminates the reference reactance from the calculation. The two methods give very similar results, and so we will compare them in that section.

The Basic macro functions for X and σ_x using the 3-V method are given on the next page. Except for a change of variables, the functions are mathematically identical to the corresponding resistance functions given in [section 2](#). Attempting to substitute variable names when coding a function call is confusing however, and so it is desirable to have separate versions.

```

Function Scalarvolt_X3V(ByVal Xref as double, Vxz as double, Vz as double, Vx as double) _
as double
'Calculates the reactive component of an impedance (X) using the 3-V scalar voltage method
IF Vx = 0 Then
Scalarvolt_X3V = 0
else
Scalarvolt_X3V = Xref*(( (Vxz^2 - Vz^2 )/ Vx^2 ) - 1) /2
end if
End function

```

```

Function SV_sigmaX3V(ByVal sigmaXref as double, Xref as double, sigmaV as double, _
OffsetV as double, Vxz as double, Vz as double, Vx as double) as double
'Calculates a standard deviation for X when the three voltage method is used.
'SigmaXref and sigmaV are input as percentages and converted into absolute uncertainties.
'OffsetV is the voltage measurement offset (or zeroing) error, in volts.
'The function returns an absolute uncertainty (in Ohms).
If Vx = 0 then
SV_sigmaX3V = 1E12
Else
sigmaXref = Abs(Xref)*sigmaXref/100
Dim sigmaVxz as double, sigmaVz as double, sigmaVx as double, Ux as double
sigmaVxz = Vxz*sigmaV/100 + OffsetV
sigmaVz = Vz*sigmaV/100 + OffsetV
sigmaVx = Vx*sigmaV/100 + OffsetV
Ux = ((Vxz^2 - Vz^2 )/ Vx^2 ) - 1
Dim derivXref as double, derivVxz as double, derivVz as double, derivVx as double
derivXref = Ux / 2
derivVxz = Vxz*Xref/Vx^2
derivVz = -Vz*Xref/Vx^2
derivVx = -Xref*((Vxz^2 - Vz^2)/Vx^3)
SV_sigmaX3V = Sqr( (sigmaXref*derivXref)^2 + (sigmaVxz*derivVxz)^2 _
+ (sigmaVz*derivVz)^2 + (sigmaVx*derivVx)^2 )
End if
End function

```

4. Reference reactance

Given the ready availability of reasonably good digital multimeters, many people have the ability to make acceptably accurate voltage and resistance measurements. Precision resistors of low series inductance are also commercially available. Equipment to make accurate reactance measurements at radio frequencies is however less common. Also, the reference capacitor (if a capacitor is used) will have some series inductance, which will cause its effective capacitance to vary with frequency; and while this can be modelled accurately over many octaves, it complicates the measurement process. Consequently, there is a general preference among users of the scalar voltage method; to eliminate the need to know the reference reactance by allowing it to be determined by the reference resistance. It would be sensible however, to find-out whether this is a good idea in terms of error-accumulation.

Referring again to [table 1](#), we can obtain the magnitude of the reference reactance using (5) / (4), i.e.;

$$\frac{X_{\text{ref}}^2}{R_{\text{ref}}^2} = \frac{|V_X|^2}{|V_R|^2}$$

Taking the square root of both sides and rearranging gives:

$$X_{\text{ref}} = \pm R_{\text{ref}} \frac{|V_X|}{|V_R|}$$

A square root has both positive and negative solutions, and so we cannot determine the sign of the reactance by this procedure. Presuming that we are using the reference component below its self-resonant frequency (SRF) however, we know the sign from the type of component. Therefore we can define a function to restore the information. For the purpose of allowing X_{ref} to vary in the process of mathematical modelling, we can write the function algebraically as:

$$\text{Sign}(X_{\text{ref}}) = \frac{X_{\text{ref}}}{|X_{\text{ref}}|}$$

This function can only take on the values +1, -1 and 0/0 (undefined). Actually, there is already built-in Sign function in Open Office Basic and other languages, and this will return 0 when the argument is 0. This avoids invalid calls to macro functions when a divide-by-zero error occurs.

So, we now have our expression for for determination of the reference reactance:

$X_{\text{ref}} = \text{Sign}(X_{\text{ref}}) R_{\text{ref}} \frac{ V_X }{ V_R }$	(4.1)
---	--------------

This combines three measurements, and if we presume them to be uncorrelated, the error function is:

$$\sigma_{X_{\text{ref}}} = \sqrt{\left(\sigma_{R_{\text{ref}}} \frac{\partial X_{\text{ref}}}{\partial R_{\text{ref}}}\right)^2 + \left(\sigma_{V_X} \frac{\partial X_{\text{ref}}}{\partial |V_X|}\right)^2 + \left(\sigma_{V_R} \frac{\partial X_{\text{ref}}}{\partial |V_R|}\right)^2}$$

where:

$$\frac{\partial X_{\text{ref}}}{\partial R_{\text{ref}}} = \text{Sign}(X_{\text{ref}}) \frac{|V_X|}{|V_R|}$$

$$\frac{\partial X_{\text{ref}}}{\partial |V_X|} = \text{Sign}(X_{\text{ref}}) \frac{R_{\text{ref}}}{|V_R|}$$

$$\frac{\partial X_{\text{ref}}}{\partial |V_R|} = -\text{Sign}(X_{\text{ref}}) \frac{R_{\text{ref}} |V_X|}{|V_R|^2}$$

(4.1u)

Notice that the Sign function is carried over into all of the derivatives. The sign is irrelevant to the error function however, because all of the error contributions will be squared, but it would be mathematically incorrect not to include it. It is shown in a different colour from the rest of the expression however, because we don't need to use it. The derivatives tell us, incidentally, that the uncertainty in X_{ref} explodes when $|V_R|$ is small.

The determination of X_{ref} from the data is simulated in the spreadsheet file [YZ_scalar_sim.ods](#) (sheet 2). Using a voltage SD of 0.5% and an SD of 0.1% for the value of R_{ref} , it returns a constant $\sigma_{X_{\text{ref}}}$ of 0.714%, regardless of the actual values of ∂X_{ref} and ∂X_{ref} . This occurs because the voltage error is assumed to be a straight percentage, i.e., offset error is ignored, but it is nevertheless realistic if the driving voltage is reasonably large.

The Basic macro error function on the next page allows an offset error to be included. If this is set to 0.01 V (typical of a 10-bit ADC with a 10 V reference), then, for $|X_{\text{ref}}|$ in the region of 50 Ω , the error is still in the 1% to 1.5% region.

The determination of X_{ref} against the reference resistance is surprisingly good; actually better than can be obtained from a typical laboratory RF bridge without spot calibration. The measurement however, does assume that there is no parasitic reactance in the reference resistance, and no parasitic resistance in the reference reactance.

Equation (4.1), of course, can be used as a substitution in other formulae in order to eliminate the need for explicit knowledge of X_{ref} .

Open Office Basic functions for X_{ref} and σ_{Xref} are given below. The sign of the reactance is not used in the error function (a standard deviation is a magnitude), but the relative signs of the derivatives are preserved internally in case of possible future modification to deal with correlated errors.

```
Function Scalarvolt_Xref(ByVal signx as integer, Rref as double, _
Vx as double, VR as double) as double
```

```
'determines Xref using Rref.
```

```
If VR = 0 then
```

```
Scalarvolt_Xref = 0
```

```
else
```

```
Scalarvolt_Xref = signx*Rref*Vx/Vr
```

```
end if
```

```
end function
```

```
Function SV_sigmaXref(ByVal sigmaRref as double, Rref as double, sigmaV as double, _
OffsetV as double, Vx as double, VR as double) as double
```

```
'calculates a standard deviation for Xref when determined against Rref.
```

```
'SigmaRref and sigmaV are input as percentages and converted into absolute uncertainties.
```

```
'OffsetV is the voltage measurement offset (or zeroing) error, in volts.
```

```
'The function returns an absolute uncertainty (in Ohms)
```

```
'sigmaXref is a magnitude, so the sign of Xref is not needed.
```

```
If VR = 0 then
```

```
SV_sigmaXref = 1E12
```

```
else
```

```
sigmaRref = Rref*sigmaRref/100
```

```
Dim sigmaVx as double, sigmaVR as double
```

```
sigmaVx = Vx*sigmaV/100 + OffsetV
```

```
sigmaVR = VR*sigmaV/100 + OffsetV
```

```
Dim derivRref as double, derivVx as double, derivVR as double
```

```
derivRref = Vx / VR
```

```
derivVx = Rref / VR
```

```
derivVR = -Rref*Vx/VR^2
```

```
SV_sigmaXref = sqr( (sigmaRref*derivRref)^2 +(sigmaVx*derivVx)^2 +(sigmaVR*derivVR)^2 )
```

```
end if
```

```
end function
```

5. Reactance: 4-V method

If we determine X_{ref} from the data using equation (4.1) and use that value to determine X by the 3-V method (equation 3.1a), then we are actually using $|V_X|$ twice. That will give rise to a correlation in the error function, and the standard deviation will not be calculated correctly. What we need to do therefore is to combine (4.1) and (3.1) and derive a new error function. Also, it means that there is no need to use X_{ref} explicitly in the calculation of X . We still need to know X_{ref} however, because it has a considerable effect on the the accuracy of measurement of both R and X , and we need to be in control of it.

Substituting (4.1) into (3.1) gives:

$$X = \text{Sign}(X_{\text{ref}}) \frac{R_{\text{ref}}}{2} \frac{|V_X|}{|V_R|} \left(\frac{|V_{XZ}|^2 - |V_Z|^2 - |V_X|^2}{|V_X|^2} \right)$$

Which, after cancellation, leaves us with:

$$X = \text{Sign}(X_{\text{ref}}) \frac{R_{\text{ref}}}{2} \left[\frac{|V_{XZ}|^2 - |V_Z|^2 - |V_X|^2}{|V_R| |V_X|} \right] \quad \text{Four voltage (4-V) method} \quad (5.1)$$

A similar expression, but without the sign, can be obtained from the analogues given in tables 1 & 2 using [(2) - (3) - (5)] / [$\sqrt{(4)} \times \sqrt{(5)}$].

The uncertainty analysis can proceed as before (although we will add 4V to the error function subscript to distinguish it from the 3-V method given earlier:

$$\sigma_{X_{4V}} = \sqrt{\left(\sigma_{R_{\text{ref}}} \frac{\partial X}{\partial R_{\text{ref}}} \right)^2 + \left(\sigma_{V_{XZ}} \frac{\partial X}{\partial |V_{XZ}|} \right)^2 + \left(\sigma_{V_Z} \frac{\partial X}{\partial |V_Z|} \right)^2 + \left(\sigma_{V_R} \frac{\partial X}{\partial |V_R|} \right)^2 + \left(\sigma_{V_X} \frac{\partial X}{\partial |V_X|} \right)^2}$$

If we call the quantity in square brackets in equation (5.1) above $U_{X_{4V}}$, then three of the derivatives can be had by inspection:

$$\frac{\partial X}{\partial R_{\text{ref}}} = \text{Sign}(X_{\text{ref}}) \frac{U_{X_{4V}}}{2} \quad [\text{dimensionless}]$$

$$\frac{\partial X}{\partial |V_{XZ}|} = \text{Sign}(X_{\text{ref}}) \frac{|V_{XZ}| R_{\text{ref}}}{|V_R| |V_X|} \quad [\text{ohms / volt}]$$

$$\frac{\partial X}{\partial |V_Z|} = -\text{Sign}(X_{\text{ref}}) \frac{|V_Z| R_{\text{ref}}}{|V_R| |V_X|} \quad [\text{ohms / volt}]$$

The remaining two however require a little more care:

$$\frac{\partial X}{\partial |V_R|} = \text{Sign}(X_{\text{ref}}) \frac{R_{\text{ref}}}{2} \left[\frac{|V_{XZ}|^2 - |V_Z|^2 - |V_X|^2}{|V_X|} \right] \left(\frac{-1}{|V_R|^2} \right)$$

Hence:

$$\frac{\partial X}{\partial |\mathbf{V}_R|} = -\text{Sign}(X_{\text{ref}}) \frac{U_{X4V} R_{\text{ref}}}{2|\mathbf{V}_R|} \quad [\text{ ohms / volt }]$$

Finally, to obtain $\partial X/\partial |\mathbf{V}_X|$ it is best to split equation (5.1) into two terms, which can be differentiated separately:

$$X = \text{Sign}(X_{\text{ref}}) \frac{R_{\text{ref}}}{2} \left[\frac{|\mathbf{V}_{XZ}|^2 - |\mathbf{V}_Z|^2}{|\mathbf{V}_R| |\mathbf{V}_X|} - \frac{|\mathbf{V}_X|}{|\mathbf{V}_R|} \right]$$

Now, noting that both differentiations produce a negative result (prior to application of the Sign operator), we get:

$$\frac{\partial X}{\partial |\mathbf{V}_X|} = \text{Sign}(X_{\text{ref}}) \frac{R_{\text{ref}}}{2} \left[\frac{|\mathbf{V}_{XZ}|^2 - |\mathbf{V}_Z|^2}{|\mathbf{V}_R| |\mathbf{V}_X|^2} + \frac{1}{|\mathbf{V}_R|} \right] \quad [\text{ ohms / volt }]$$

As in the previous section, the sign of the reference reactance makes no difference to the error function, but it is preserved in the derivation for reasons of mathematical rigour.

The collected formulae are:

$X = \text{Sign}(X_{\text{ref}}) \frac{U_{X4V} R_{\text{ref}}}{2} \quad \text{where:} \quad U_{X4V} = \frac{ \mathbf{V}_{XZ} ^2 - \mathbf{V}_Z ^2 - \mathbf{V}_X ^2}{ \mathbf{V}_R \mathbf{V}_X }$	<p>(5.1a) 4-V</p>
$\sigma_{X4V} = \sqrt{\left(\sigma_{R_{\text{ref}}} \frac{\partial X}{\partial R_{\text{ref}}} \right)^2 + \left(\sigma_{V_{XZ}} \frac{\partial X}{\partial \mathbf{V}_{XZ} } \right)^2 + \left(\sigma_{V_Z} \frac{\partial X}{\partial \mathbf{V}_Z } \right)^2 + \left(\sigma_{V_R} \frac{\partial X}{\partial \mathbf{V}_R } \right)^2 + \left(\sigma_{V_X} \frac{\partial X}{\partial \mathbf{V}_X } \right)^2}$ $\frac{\partial X}{\partial R_{\text{ref}}} = \text{Sign}(X_{\text{ref}}) \frac{U_{X4V}}{2}$ $\frac{\partial X}{\partial \mathbf{V}_{XZ} } = \text{Sign}(X_{\text{ref}}) \frac{ \mathbf{V}_{XZ} R_{\text{ref}}}{ \mathbf{V}_R \mathbf{V}_X }$ $\frac{\partial X}{\partial \mathbf{V}_Z } = -\text{Sign}(X_{\text{ref}}) \frac{ \mathbf{V}_Z R_{\text{ref}}}{ \mathbf{V}_R \mathbf{V}_X }$ $\frac{\partial X}{\partial \mathbf{V}_R } = -\text{Sign}(X_{\text{ref}}) \frac{U_{X4V} R_{\text{ref}}}{2 \mathbf{V}_R }$ $\frac{\partial X}{\partial \mathbf{V}_X } = \text{Sign}(X_{\text{ref}}) \frac{R_{\text{ref}}}{2} \left[\frac{ \mathbf{V}_{XZ} ^2 - \mathbf{V}_Z ^2}{ \mathbf{V}_R \mathbf{V}_X ^2} + \frac{1}{ \mathbf{V}_R } \right]$	<p>(5.1u)</p>

OO Basic functions for the calculation of X and σ_X by the four-voltage method are shown below.

```
Function Scalarvolt_X4V(ByVal signx as integer, Rref as double, _
Vxz as double, Vz as double, Vx as double, VR as double) as double
'Calculates the reactive component of an impedance (X) using the 4-V scalar voltage method
IF VR = 0 or Vx = 0 Then
Scalarvolt_X4V = 0
else
Scalarvolt_X4V = signx*Rref*(Vxz^2 - Vz^2 - vx^2)/(2*VR*Vx)
end if
End function
```

```
Function SV_sigmaX4V(ByVal sigmaRref as double, Rref as double, sigmaV as double, _
offsetV as double, Vxz as double, Vz as double, Vx as double, VR as double) as double
'Calculates a standard deviation for X when the four voltage method is used.
'SigmaRref and sigmaV are input as percentages and converted into absolute uncertainties.
'offsetV is the voltage measurement offset (or zeroing) error, in volts.
'The function returns an absolute uncertainty (in Ohms).
'sigmaX is a magnitude, and so the sign of Xref is not needed.
If Vx = 0 or VR = 0 then
SV_sigmaX4V = 1E12
Else
sigmaRref = Rref*sigmaRref/100
Dim sigmaVxz as double, sigmaVz as double, sigmaVx as double, sigmaVR as double
Dim Ux4v as double, derivRref as double
Dim derivVxz as double, derivVz as double, derivVx as double, derivVR as double
sigmaVxz = Vxz*sigmaV/100 + offsetV
sigmaVz = Vz*sigmaV/100 + offsetV
sigmaVx = Vx*sigmaV/100 + offsetV
sigmaVR = VR*sigmaV/100 + offsetV
Ux4v = ( Vxz^2 - Vz^2 - Vx^2 ) / ( VR*Vx )
derivRref = Ux4v / 2
derivVxz = Vxz*Rref/(Vx*VR)
derivVz = -Vz*Rref/(Vx*VR)
derivVR = -Ux4v*Rref/(2*VR)
derivVx = 0.5*Rref*( ( Vxz^2-Vz^2 )/( VR*Vx^2 )) +1/VR )
SV_sigmaX4V = Sqr( (sigmaRref*derivRref)^2 + (sigmaVxz*derivVxz)^2 _
+ (sigmaVz*derivVz)^2 + (sigmaVR*derivVR)^2 + (sigmaVx*derivVx)^2 )
End if
End function
```

An error simulation worksheet for the four-voltage method for reactance is given in the spreadsheet file [YZ_scalar_sim.ods](#) (sheet 3). Once again, the working is checked against a separately coded macro function. The error function for the 3-V method (calculated using the macro given in [section 3](#)) is also plotted on the graphs for comparison.

Recall that the 3-V method requires a reference reactance value as input; and so for the purpose of the comparison, $\sigma_{X_{\text{ref}}}$ is set to the value that would be obtained from a determination of X_{ref} carried out as in [section 4](#) (i.e., computed from R_{ref} and the voltages) and using the same measurement conditions as for X . Since the uncertainty of X_{ref} is then correlated with the uncertainty contributions used in the error calculations for X , we expect there to be a difference in the reported standard deviation depending on whether the reference reactance is used explicitly (3-V method) or implicitly (4-V method). What we are testing therefore is: given equal uncertainty between an independent determination of X_{ref} (obtained using a separate measuring device) or an implicit determination, which method will give the best result? A difference is seen in the graphs below but it is fairly small. Generally, the 4-V method is slightly better than the 3-V method, except when the magnitude of the reference reactance is somewhat greater than that of the unknown reactance and of opposite sign. The 3-V calculation assumes however, that a determination of X_{ref} , at least as good as can be obtained from the data, is available from another source. A very accurate model for X_{ref} might lead us to use the 3-V method, but the 4-V method is otherwise to be preferred on grounds of accuracy and convenience.

The simulations shown in the graph on the right are carried out for an 'unknown' impedance with a reactive component of $+50 \Omega$. The error parameters are:

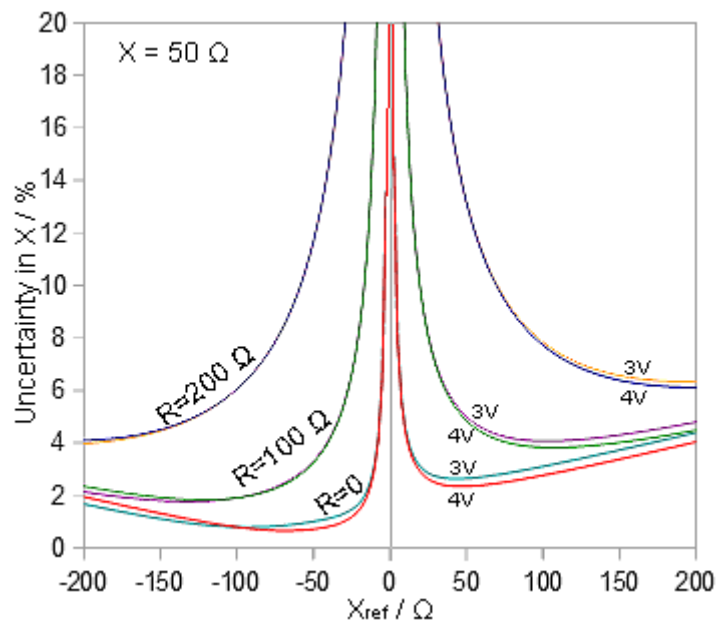
$$\sigma_{R_{\text{ref}}} = 0.1\%$$

$$\sigma_V = 0.5\%$$

and for the 3-V calculation:

$$\sigma_{X_{\text{ref}}} = 0.714\%$$

The X-axis shows the effect of varying the reference reactance. Obviously, $\sigma_X \rightarrow \infty$ when $X_{\text{ref}} \rightarrow 0$. Also there is a conjugate reactance effect, in that the error is less when the reference reactance is of opposite sign to the unknown reactance. If the sign of the unknown reactance is reversed, the shape of the graph is reflected about $X_{\text{ref}} = 0$. Notice however, that the minimum error does not occur when $X_{\text{ref}} = -X$. This is because the minimum is skewed by the infinity at $X_{\text{ref}} = 0$.



The separate curves show that the measurement error is dependent on the resistive component of the unknown impedance. Best accuracy is obtained when $R = 0$, but the error incurred when measuring impedances with a larger resistive component can be ameliorated by making $|X_{\text{ref}}| > R$.

Note that, when using a straight percentage error for voltages, the calculations give the same result regardless of R_{ref} (provided that it is > 0), because there is no penalty associated with the measurement of very small voltages. Consequently, adjusting R_{ref} makes no difference to the simulation (although we need to develop a few volts across it in practice).

6. Impedance magnitude

Impedance magnitude is defined as:

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}$$

This can be obtained from [table 1](#) using (3) / (4), i.e.;

$$\frac{|\mathbf{V}_Z|^2}{|\mathbf{V}_R|^2} = \frac{\sqrt{R^2 + X^2}}{R_{\text{ref}}}$$

Which gives:

$ \mathbf{Z} = R_{\text{ref}} \frac{ \mathbf{V}_Z }{ \mathbf{V}_R }$	(6.1)
---	--------------

The uncorrelated error function is:

$\sigma_Z = \sqrt{\left(\sigma_{R_{\text{ref}}} \frac{\partial Z}{\partial R_{\text{ref}}}\right)^2 + \left(\sigma_{V_Z} \frac{\partial Z}{\partial \mathbf{V}_Z }\right)^2 + \left(\sigma_{V_R} \frac{\partial Z}{\partial \mathbf{V}_R }\right)^2}$	
with derivatives:	
$\frac{\partial Z}{\partial R_{\text{ref}}} = \frac{ \mathbf{V}_Z }{ \mathbf{V}_R }$	(6.1u)
$\frac{\partial Z}{\partial \mathbf{V}_Z } = \frac{R_{\text{ref}}}{ \mathbf{V}_R }$	
$\frac{\partial Z}{\partial \mathbf{V}_R } = -R_{\text{ref}} \frac{ \mathbf{V}_Z }{ \mathbf{V}_R ^2}$	

A simulation of the calculation is given in the spreadsheet [YZ_scalar_sim.ods](#) (sheet 1). Since the error function is the same as that for the reference reactance determination, except for a change of variables, it should not be surprising that the accuracy obtained is the same, i.e. constant for a given simple percentage σ_V and $\sigma_{R_{\text{ref}}}$. Using 0.5% and 0.1% as before, we obtain $\sigma_Z = 0.714\%$. The scalar voltage method is evidently a fairly good choice for the determination of impedance magnitude.

Basic macros for $|\mathbf{Z}|$ and σ_Z are given on the next page.

```

Function Scalarvolt_ModZ(ByVal Rref as double, Vz as double, VR as double) as double
'Calculates |Z| by comparing the impedance against a reference resistance.
If VR = 0 then
Scalarvolt_ModZ = 0
else
Scalarvolt_ModZ = Rref*Vz/VR
end if
End Function

```

```

Function SV_sigmaZ(ByVal sigmaRref as double, Rref as double, sigmaV as double, _
offsetV as double, VZ as double, VR as double) as double
'Calculates a standard deviation for |Z|
'SigmaRref and sigmaV are input in % and converted into absolute uncertainties.
'offsetV is the voltage measurement offset (or zeroing) error, in volts.
'The function returns an absolute uncertainty (in Ohms).
If VR = 0 then
SV_sigmaZ = 1E12
Else
sigmaRref = Rref*sigmaRref/100
Dim sigmaVZ as double, sigmaVR as double
Dim derivRref as double, derivVZ as double, derivVR as double
sigmaVZ = VZ*sigmaV/100 + offsetV
sigmaVR = VR*sigmaV/100 + offsetV
derivRref = VZ / VR
derivVZ = Rref / VR
derivVR = -Rref*VZ/VR^2
SV_sigmaZ = Sqr( (sigmaRref*derivRref)^2 +(sigmaVz*derivVZ)^2 +(sigmaVR*derivVR)^2 )
End if
End Function

```

7. Phase tangent and Q

The phase angle of an impedance $R+jX$ is given by:

$$\phi = \arctan\left(\frac{X}{R}\right)$$

This is the quantity that must be brought to zero during the reactance cancelling step of an impedance-matching operation. However, for any impedance matching process, be it manual or automatic, it makes little difference whether we use the cyclometric (inverse trig.) function above, or whether we use the quantity:

$$\tan(\phi) = \frac{X}{R}$$

Note also that the Q of an impedance is defined as:

$$Q = \frac{|X|}{R}$$

So if we determine $\tan\phi$, we get Q gratis. Also observe that if we calculate a standard deviation for $\tan\phi$, we can call it σ_Q without any loss of generality.

The most obvious way in which to determine $\tan\phi$ from the raw voltage-measurement data would be to take an expression for X and divide it by an expression for R. It is also obvious that, since we have two expressions for X, there will be two ways of calculating X/R, one using X_{ref} explicitly, and one using it implicitly. It is a little more informative however, to start with the analogs in [table 2](#). Using (2) - (3) - (5) / [(1) - (2) - (4)] we get:

$$\frac{|V_{xz}|^2 - |V_z|^2 - |V_x|^2}{|V_s|^2 - |V_{xz}|^2 - |V_R|^2} = \frac{X X_{ref}}{R R_{ref}}$$

Hence:

$$\frac{X}{R} = \frac{R_{ref}}{X_{ref}} \left[\frac{|V_{xz}|^2 - |V_z|^2 - |V_x|^2}{|V_s|^2 - |V_{xz}|^2 - |V_R|^2} \right] \quad \text{Explicit reference method} \quad (7.1)$$

This requires knowledge of both R_{ref} and X_{ref} , and we will therefore refer to it as the 'explicit reference method'. As in the case of the 3-V formula for X, it also gives the sign of X/R provided that we know the sign of X_{ref} .

As was discussed in [section 4](#), The reference reactance can be eliminated by using the substitution (4.1):

$$X_{ref} = \text{Sign}(X_{ref}) R_{ref} \frac{|V_x|}{|V_R|}$$

Where $\text{Sign}(X_{ref})$ is the sign of the reference reactance (+1 or -1) determined from knowledge of the type of component in use (-1 for a capacitor).

Hence:

$$\frac{X}{R} = \text{Sign}(X_{\text{ref}}) \frac{|V_R|}{|V_X|} \left[\frac{|V_{XZ}|^2 - |V_Z|^2 - |V_X|^2}{|V_S|^2 - |V_{XZ}|^2 - |V_R|^2} \right] \quad \text{Implicit reference method} \quad (7.2)$$

Now the calculation does not require the value of either of the references components, and so we will refer to it as the 'implicit reference method'. The choice of reference values however will still have an effect on the accuracy of the result.

In order to obtain error functions for either of the two methods, it will be necessary to differentiate the corresponding expression with respect to every one of its parameters. Some reduction of duplicated effort will therefore be obtained by preserving the similarities between the the two methods. First we will define the quantity that appears in square brackets in both expressions as:

$$U_Q = \left[\frac{|V_{XZ}|^2 - |V_Z|^2 - |V_X|^2}{|V_S|^2 - |V_{XZ}|^2 - |V_R|^2} \right] \quad (7.3)$$

Then, the formula for $\tan\phi$ by the explicit reference method becomes:

$$\frac{X}{R} = U_Q \frac{R_{\text{ref}}}{X_{\text{ref}}} \quad (7.1a)$$

and the formula for the implicit reference method becomes:

$$\frac{X}{R} = \text{Sign}(X_{\text{ref}}) U_Q \frac{|V_R|}{|V_X|} \quad (7.2a)$$

But note that a standard deviation is a magnitude. Consequently, we don't need to include the sign when deriving the error function. So, we can eliminate the sign operator from (7.2a) by starting with an expression for Q , but we will use the same approach in both cases to preserve the equivalence:

$$Q = \left| U_Q \frac{R_{\text{ref}}}{X_{\text{ref}}} \right| \quad \text{Explicit reference}$$

$$Q = |U_Q| \frac{|V_R|}{|V_X|} \quad \text{Implicit reference}$$

Although it is not obvious at this stage, U_Q can be either positive or negative. Consequently, since Q is positive by convention, one of the expressions requires us to work with the magnitude of U_Q ; and in the process of obtaining derivatives, there is a difference in the propagation of algebraic signs. Both also force us to compute magnitudes unnecessarily during differentiation, since the sign of Q has no effect on the error function. We can however, circumvent that issue by defining a quantity q as follows:

$$q = U_Q \frac{R_{\text{ref}}}{|X_{\text{ref}}|} \quad \text{Explicit reference} \quad (7.4)$$

$$q = U_Q \frac{|V_R|}{|V_X|} \quad \text{Implicit reference} \quad (7.5)$$

The two functions now allow the sign of q to be dictated by U_Q . Q (which we do not need) is equal to the magnitude of q ; but q itself can be either positive or negative.

7a. Error function for explicit reference method

As explained in the previous section, the error function for X/R is the same as the error function for Q , but it is convenient to derive it from a quantity q (defined such that $Q = |q|$) as given by equation (7.4), i.e.; $q = U_Q R_{\text{ref}} / |X_{\text{ref}}|$. U_Q is a function of all five measured voltages, as can be seen by examining equation (7.3). Hence the error function for the explicit reference method (indicated by a small x in the subscript) can be written:

$$\begin{aligned} \sigma_{Q_x}^2 = & \left(\sigma_{R_{\text{ref}}} \frac{\partial q}{\partial R_{\text{ref}}} \right)^2 + \left(\sigma_{X_{\text{ref}}} \frac{\partial q}{\partial |X_{\text{ref}}|} \right)^2 \\ & + \left(\sigma_{V_{XZ}} \frac{\partial q}{\partial |V_{XZ}|} \right)^2 + \left(\sigma_{V_Z} \frac{\partial q}{\partial |V_Z|} \right)^2 + \left(\sigma_{V_X} \frac{\partial q}{\partial |V_X|} \right)^2 + \left(\sigma_{V_S} \frac{\partial q}{\partial |V_S|} \right)^2 + \left(\sigma_{V_R} \frac{\partial q}{\partial |V_R|} \right)^2 \end{aligned}$$

The first two derivatives are straightforward:

$$\frac{\partial q}{\partial R_{\text{ref}}} = \frac{U_q}{|X_{\text{ref}}|} \quad [\text{Siemens}]$$

$$\frac{\partial q}{\partial |X_{\text{ref}}|} = -U_q \frac{R_{\text{ref}}}{X_{\text{ref}}^2} \quad [\text{Siemens}]$$

The dimensions of these derivatives are Siemens (i.e., $[\Omega^{-1}]$) because Q (or q , or X/R) is a dimensionless ratio, and so the product of the derivative and the contributing uncertainty (in ohms) must be dimensionless.

$|V_{XZ}|$ appears in both the numerator and denominator of U_Q , and so to obtain $\partial Q / \partial |V_{XZ}|$ we can use the quotient rule:

$$\frac{d \left(\frac{u}{w} \right)}{d x} = \frac{w \frac{d u}{d x} - u \frac{d w}{d x}}{w^2}$$

so if u is the numerator of U_Q and w is the denominator, we have:

$$u = |V_{XZ}|^2 - |V_Z|^2 - |V_X|^2 \quad [\text{volts}^2]$$

$$w = |V_S|^2 - |V_{XZ}|^2 - |V_R|^2 \quad [\text{volts}^2]$$

Thus:

$$\frac{\partial u}{\partial |\mathbf{V}_{xz}|} = 2|\mathbf{V}_{xz}| \quad , \quad \frac{\partial w}{\partial |\mathbf{V}_{xz}|} = -2|\mathbf{V}_{xz}| \quad ,$$

$$\frac{\partial (u/w)}{\partial |\mathbf{V}_{xz}|} = \frac{2|\mathbf{V}_{xz}| w + 2|\mathbf{V}_{xz}| u}{w^2} = \frac{2|\mathbf{V}_{xz}| (w + u)}{w^2}$$

Hence:

$$\frac{\partial q}{\partial |\mathbf{V}_{xz}|} = 2 \frac{R_{ref}}{|X_{ref}|} |\mathbf{V}_{xz}| \frac{(w + u)}{w^2} \quad [\text{volts} \times \text{volts}^2 / \text{volts}^4 = \text{volts}^{-1}]$$

w and u are going to appear again, so we might as well evaluate them during the calculation process (i.e., there is no point in writing out the expression in full).

$|\mathbf{V}_z|$ and $|\mathbf{V}_x|$ only appear in the numerator of U_Q and so the next two derivatives are straightforward:

$$\frac{\partial q}{\partial |\mathbf{V}_z|} = -2 \frac{R_{ref}}{|X_{ref}|} \frac{|\mathbf{V}_z|}{w} \quad [\text{volts} / \text{volts}^2 = \text{volts}^{-1}]$$

$$\frac{\partial q}{\partial |\mathbf{V}_x|} = -2 \frac{R_{ref}}{|X_{ref}|} \frac{|\mathbf{V}_x|}{w} \quad [\text{volts} / \text{volts}^2 = \text{volts}^{-1}]$$

$|\mathbf{V}_s|$ and $|\mathbf{V}_R|$ only appear in the denominator of U_Q and so we need to use the 'function of a function' rule:

$$\text{if } y = f(w) \text{ and } w = f(v) \text{ then } \frac{dy}{dv} = \frac{dy}{dw} \cdot \frac{dw}{dv}$$

$$\text{so if } y = \frac{1}{w} \text{ then } \frac{dy}{dw} = \frac{-1}{w^2} \quad ; \text{ and if } w = a + bv^2 \text{ then } \frac{dw}{dv} = 2bv \quad .$$

$$\text{Hence if } y = \frac{1}{(a + bv^2)} \text{ then } \frac{dy}{dv} = \frac{-2bv}{w^2} \quad .$$

Thus, putting $v = |\mathbf{V}_s|$ and $b = +1$:

$$\frac{\partial q}{\partial |\mathbf{V}_s|} = -2 \frac{R_{ref}}{|X_{ref}|} |\mathbf{V}_s| \frac{u}{w^2} \quad [\text{volts} \times \text{volts}^2 / \text{volts}^4 = \text{volts}^{-1}]$$

and, putting $v = |\mathbf{V}_R|$ and $b = -1$:

$$\frac{\partial q}{\partial |\mathbf{V}_R|} = 2 \frac{R_{ref}}{|X_{ref}|} |\mathbf{V}_R| \frac{u}{w^2} \quad [\text{volts} \times \text{volts}^2 / \text{volts}^4 = \text{volts}^{-1}]$$

The collected formulae are:

$\tan \phi = \frac{X}{R} = U_Q \frac{R_{\text{ref}}}{X_{\text{ref}}} \quad \text{where} \quad U_Q = \frac{u}{w},$ $u = \mathbf{V}_{\text{xz}} ^2 - \mathbf{V}_z ^2 - \mathbf{V}_x ^2 \quad \text{and} \quad w = \mathbf{V}_s ^2 - \mathbf{V}_{\text{xz}} ^2 - \mathbf{V}_R ^2$	(7.6) tanφ explicit reference method
$q = U_Q \frac{R_{\text{ref}}}{ X_{\text{ref}} }, \quad \sigma_{Qx} = \sqrt{\sigma_{Qx}^2},$ $\sigma_{Qx}^2 = \left(\sigma_{R_{\text{ref}}} \frac{\partial q}{\partial R_{\text{ref}}} \right)^2 + \left(\sigma_{X_{\text{ref}}} \frac{\partial q}{\partial X_{\text{ref}} } \right)^2$ $+ \left(\sigma_{V_{\text{xz}}} \frac{\partial q}{\partial \mathbf{V}_{\text{xz}} } \right)^2 + \left(\sigma_{V_z} \frac{\partial q}{\partial \mathbf{V}_z } \right)^2 + \left(\sigma_{V_x} \frac{\partial q}{\partial \mathbf{V}_x } \right)^2 + \left(\sigma_{V_s} \frac{\partial q}{\partial \mathbf{V}_s } \right)^2 + \left(\sigma_{V_R} \frac{\partial q}{\partial \mathbf{V}_R } \right)^2$ <p>Derivatives:</p> $\frac{\partial q}{\partial R_{\text{ref}}} = \frac{U_q}{ X_{\text{ref}} }$ $\frac{\partial q}{\partial X_{\text{ref}} } = -U_q \frac{R_{\text{ref}}}{X_{\text{ref}}^2}$ $\frac{\partial q}{\partial \mathbf{V}_{\text{xz}} } = 2 \frac{R_{\text{ref}}}{ X_{\text{ref}} } \mathbf{V}_{\text{xz}} \frac{(w+u)}{w^2}$ $\frac{\partial q}{\partial \mathbf{V}_z } = -2 \frac{R_{\text{ref}}}{ X_{\text{ref}} } \frac{ \mathbf{V}_z }{w}$ $\frac{\partial q}{\partial \mathbf{V}_x } = -2 \frac{R_{\text{ref}}}{ X_{\text{ref}} } \frac{ \mathbf{V}_x }{w}$ $\frac{\partial q}{\partial \mathbf{V}_s } = -2 \frac{R_{\text{ref}}}{ X_{\text{ref}} } \mathbf{V}_s \frac{u}{w^2}$ $\frac{\partial q}{\partial \mathbf{V}_R } = 2 \frac{R_{\text{ref}}}{ X_{\text{ref}} } \mathbf{V}_R \frac{u}{w^2}$	(7.6u)

Basic macros for these calculations are shown on the next page. A simulation worksheet is given in the spreadsheet file [YZ_scalar_sim.ods](#) (sheet 4). In the worksheet, the method is verified against the Basic macro, and compared with the implicit reference method (to be discussed next).

```

Function SVx_Tanphi(ByVal Rref as double, Xref as double, _
Vxz as double, Vz as double, Vx as double, Vs as double, VR as double) as double
'Calculates Tan(phi) = X/R by the explicit reference method.
Dim w as double, UQ as double
w = Vs^2 - Vxz^2 - VR^2
IF Xref = 0 or w = 0 Then
SVx_Tanphi = 0
else
UQ = (Vxz^2 - Vz^2 - Vx^2) / w
SVx_Tanphi = UQ * Rref / Xref
end if
end function

```

```

Function SV_sigmaQx(ByVal Rref as double, sigmaRref as double, _
Xref as double, sigmaXref as double, sigmaV as double, offsetV as double, _
Vxz as double, Vz as double, Vx as double, Vs as double, VR as double) as double
'Calculates sigmaQ using the explicit reference method.
Dim u as double, w as double, UQ as double
w = Vs^2 - Vxz^2 - VR^2
IF Xref = 0 or w = 0 Then
SV_sigmaQx = 1E12
else
Dim sigmaVxz as double, sigmaVz as double, sigmaVx as double
Dim sigmaVs as double, sigmaVR as double
sigmaVxz = Vxz*sigmaV/100 + offsetV
sigmaVz = Vz*sigmaV/100 + offsetV
sigmaVx = Vx*sigmaV/100 + offsetV
sigmaVs = Vs*sigmaV/100 + offsetV
sigmaVR = VR*sigmaV/100 + offsetV
Dim derRref as double, derXref as double, derVxz as double, derVz as double
Dim derVx as double, derVs as double, derVR as double, RbyX as double
u = Vxz^2 - Vz^2 - Vx^2
UQ = u / w
Xref = abs(Xref)
derRref = UQ/Xref
derXref = -UQ*Rref/Xref^2
RbyX = Rref / Xref
derVxz = 2*RbyX*Vxz*(w+u)/w^2
derVz = -2*RbyX*Vz/w
derVx = -2*RbyX*Vx/w
derVs = -2*RbyX*Vs*u/w^2
derVR = 2*RbyX*VR*u/w^2
SV_sigmaQx = sqr( (sigmaRref*derRref)^2 +(sigmaXref*derXref)^2 +(sigmaVxz*derVxz)^2 _
+(sigmaVz*derVz)^2 +(sigmaVx*derVx)^2 +(sigmaVs*derVs)^2 +(sigmaVR*derVR)^2 )
end if
end function

```

7b. Error function for implicit reference method

In deriving the error function for the implicit reference method, we will preserve the definitions of U_Q , u and w from the previous section. Thus, from equation (7.5) we have:

$$q = U_Q \frac{|\mathbf{V}_R|}{|\mathbf{V}_X|} = \frac{|\mathbf{V}_R|}{|\mathbf{V}_X|} \frac{u}{w}$$

and the error function (with i , for implicit, added to the subscript) becomes:

$$\sigma_{Qi} = \sqrt{\left(\sigma_{vXZ} \frac{\partial q}{\partial |\mathbf{V}_{XZ}|}\right)^2 + \left(\sigma_{vZ} \frac{\partial q}{\partial |\mathbf{V}_Z|}\right)^2 + \left(\sigma_{vX} \frac{\partial q}{\partial |\mathbf{V}_X|}\right)^2 + \left(\sigma_{vS} \frac{\partial q}{\partial |\mathbf{V}_S|}\right)^2 + \left(\sigma_{vR} \frac{\partial q}{\partial |\mathbf{V}_R|}\right)^2}$$

The derivatives must also be redefined, although in the three instances where $|\mathbf{V}_R|$ and $|\mathbf{V}_X|$ are held constant, the difference is only a matter of substituting $|\mathbf{V}_R| / |\mathbf{V}_X|$ in place of $|R_{ref} / X_{ref}|$. Thus:

$$\frac{\partial q}{\partial |\mathbf{V}_{XZ}|} = 2 \frac{|\mathbf{V}_R|}{|\mathbf{V}_X|} |\mathbf{V}_{XZ}| \frac{(w+u)}{w^2}$$

$$\frac{\partial q}{\partial |\mathbf{V}_Z|} = -2 \frac{|\mathbf{V}_R|}{|\mathbf{V}_X|} \frac{|\mathbf{V}_Z|}{w}$$

$$\frac{\partial q}{\partial |\mathbf{V}_S|} = -2 \frac{|\mathbf{V}_R|}{|\mathbf{V}_X|} |\mathbf{V}_S| \frac{u}{w^2}$$

The remaining two however are rather more difficult.

To find $\partial q / \partial |\mathbf{V}_R|$, we can start by regrouping the expression for q as:

$$q = \frac{u}{|\mathbf{V}_X|} \frac{|\mathbf{V}_R|}{w}$$

The first fraction contains the quantities held constant for this differentiation, and the second contains the functions of the variable. Thus the expression is now in the form:

$$q = k \frac{z(v)}{w(v)} \quad ; \quad \text{so that} \quad \frac{dq}{dv} = k \frac{\left(w \frac{dz}{dv} - z \frac{dw}{dv} \right)}{w^2}$$

where $z = v$ giving $dz/dv = 1$ and $w = a - v^2$ giving $dw/dv = -2v$.

Thus: $\frac{dq}{dv} = \frac{k(w+2v^2)}{w^2}$ and so:

$$\frac{\partial q}{\partial |\mathbf{V}_R|} = \frac{u}{|\mathbf{V}_X|} \left[\frac{1}{w} + \frac{2|\mathbf{V}_R|^2}{w^2} \right] \quad [\text{volts} \times \text{volts}^{-2} = \text{volts}^{-1}]$$

To find $\partial q / \partial |\mathbf{V}_x|$, we can regroup the expression for q as:

$$q = \frac{|\mathbf{V}_R|}{w} \frac{u}{|\mathbf{V}_x|}$$

where the variables are in the second fraction. The expression is now in the form:

$$q = k \frac{u(v)}{y(v)} \quad ; \text{ so that } \quad \frac{dq}{dv} = k \frac{\left(y \frac{du}{dv} - u \frac{dy}{dv} \right)}{y^2}$$

where $y = v$ giving $dy/dv = 1$ and $u = a - v^2$ giving $du/dv = -2v$

$$\text{therefore: } \quad \frac{dq}{dv} = \frac{k(-2v^2 - u)}{v^2} = k \left(-2 - \frac{u}{v^2} \right) \quad \text{and so:}$$

$$\frac{\partial q}{\partial |\mathbf{V}_x|} = -\frac{|\mathbf{V}_R|}{w} \left(2 + \frac{u}{|\mathbf{V}_x|^2} \right) \quad [\text{volts}^{-1}]$$

We can also check this differentiation by performing it in a different way. Expanding u we get:

$$q = \frac{|\mathbf{V}_R|}{w} \left(\frac{|\mathbf{V}_{xz}|^2 - |\mathbf{V}_z|^2 - |\mathbf{V}_x|^2}{|\mathbf{V}_x|} \right)$$

and dividing through gives:

$$q = \frac{|\mathbf{V}_R|}{w} \left(\frac{|\mathbf{V}_{xz}|^2 - |\mathbf{V}_z|^2}{|\mathbf{V}_x|} - |\mathbf{V}_x| \right)$$

This can be differentiated directly:

$$\frac{\partial q}{\partial |\mathbf{V}_x|} = \frac{|\mathbf{V}_R|}{w} \left(-\frac{|\mathbf{V}_{xz}|^2 - |\mathbf{V}_z|^2}{|\mathbf{V}_x|^2} - 1 \right)$$

Taking out the minus sign and putting both terms on a common denominator gives:

$$\frac{\partial q}{\partial |\mathbf{V}_x|} = -\frac{|\mathbf{V}_R|}{w} \left(\frac{|\mathbf{V}_{xz}|^2 - |\mathbf{V}_z|^2 + |\mathbf{V}_x|^2}{|\mathbf{V}_x|^2} \right)$$

Then using the substitution: $|\mathbf{V}_x|^2 = 2|\mathbf{V}_x|^2 - |\mathbf{V}_x|^2$ we get:

$$\frac{\partial q}{\partial |\mathbf{V}_x|} = -\frac{|\mathbf{V}_R|}{w} \left(\frac{|\mathbf{V}_{xz}|^2 - |\mathbf{V}_z|^2 - |\mathbf{V}_x|^2 + 2|\mathbf{V}_x|^2}{|\mathbf{V}_x|^2} \right) = -\frac{|\mathbf{V}_R|}{w} \left(\frac{u}{|\mathbf{V}_x|^2} + 2 \right)$$

which is the same as above.

The collected formulae are:

$\tan \phi = \frac{X}{R} = \text{Sign}(X_{\text{ref}}) U_Q \frac{ \mathbf{V}_R }{ \mathbf{V}_X }$ <p>where</p> $U_Q = \frac{u}{w}$ $u = \mathbf{V}_{XZ} ^2 - \mathbf{V}_Z ^2 - \mathbf{V}_X ^2$ $w = \mathbf{V}_S ^2 - \mathbf{V}_{XZ} ^2 - \mathbf{V}_R ^2$	<p>(7.7)</p> <p>tanφ implicit reference method</p>
$q = U_Q \frac{ \mathbf{V}_R }{ \mathbf{V}_X }$ $\sigma_{q_i} = \sqrt{\left(\sigma_{v_{XZ}} \frac{\partial q}{\partial \mathbf{V}_{XZ} }\right)^2 + \left(\sigma_{v_Z} \frac{\partial q}{\partial \mathbf{V}_Z }\right)^2 + \left(\sigma_{v_X} \frac{\partial q}{\partial \mathbf{V}_X }\right)^2 + \left(\sigma_{v_S} \frac{\partial q}{\partial \mathbf{V}_S }\right)^2 + \left(\sigma_{v_R} \frac{\partial q}{\partial \mathbf{V}_R }\right)^2}$ <p>with derivatives:</p> $\frac{\partial q}{\partial \mathbf{V}_{XZ} } = 2 \frac{ \mathbf{V}_R }{ \mathbf{V}_X } \mathbf{V}_{XZ} \frac{(w+u)}{w^2}$ $\frac{\partial q}{\partial \mathbf{V}_Z } = -2 \frac{ \mathbf{V}_R }{ \mathbf{V}_X } \frac{ \mathbf{V}_Z }{w}$ $\frac{\partial q}{\partial \mathbf{V}_X } = -\frac{ \mathbf{V}_R }{w} \left(2 + \frac{u}{ \mathbf{V}_X ^2}\right)$ $\frac{\partial q}{\partial \mathbf{V}_S } = -2 \frac{ \mathbf{V}_R }{ \mathbf{V}_X } \mathbf{V}_S \frac{u}{w^2}$ $\frac{\partial q}{\partial \mathbf{V}_R } = \frac{u}{ \mathbf{V}_X } \left[\frac{1}{w} + \frac{2 \mathbf{V}_R ^2}{w^2}\right]$	<p>(7.7u)</p>

Basic macros are shown on the next page.


```

Function SVi_Tanphi(ByVal signx as integer, Vxz as double, Vz as double, Vx as double, _
Vs as double, VR as double) as double
'Calculates Tan(phi) = X/R by the implicit reference method.
Dim w as double, UQ as double
w = Vs^2 - Vxz^2 - VR^2
if Vx = 0 or w = 0 then
  SVi_Tanphi = 0
else
  UQ = (Vxz^2 - Vz^2 - Vx^2)/w
  SVi_Tanphi = signx*uQ*VR/Vx
end if
end function

```

```

Function SV_sigmaQi(ByVal sigmaV as double, offsetV as double, _
Vxz as double, Vz as double, Vx as double, Vs as double, VR as double) as double
'Calculates sigmaQ using the implicit reference method.
Dim u as double, w as double
w = Vs^2 - Vxz^2 - VR^2
if Vx = 0 or w = 0 Then
  SV_sigmaQi = 1E12
else
  u = Vxz^2 - Vz^2 - Vx^2
  Dim sigmaVxz as double, sigmaVz as double, sigmaVx as double
  Dim sigmaVs as double, sigmaVR as double
  sigmaVxz = Vxz*sigmaV/100 + offsetV
  sigmaVz = Vz*sigmaV/100 + offsetV
  sigmaVx = Vx*sigmaV/100 + offsetV
  sigmaVs = Vs*sigmaV/100 + offsetV
  sigmaVR = VR*sigmaV/100 + offsetV
  Dim VRbyVx as double, derVxz as double, derVz as double
  Dim derVx as double, derVs as double, derVR as double
  VRbyVx = VR / Vx
  derVxz = 2*VRbyVx*Vxz*(w+u)/w^2
  derVz = -2*VRbyVx*Vz/w
  derVx = -(VR/w)*(2+u/Vx^2)
  derVs = -2*VRbyVx*Vs*u/w^2
  derVR = (u/Vx)*((1/w) + 2*VR^2/w^2)
  SV_sigmaQi = sqr( (sigmaVxz*derVxz)^2 +(sigmaVz*derVz)^2 +(sigmaVx*derVx)^2 _
+(sigmaVs*derVs)^2 +(sigmaVR*derVR)^2 )
end if
end function

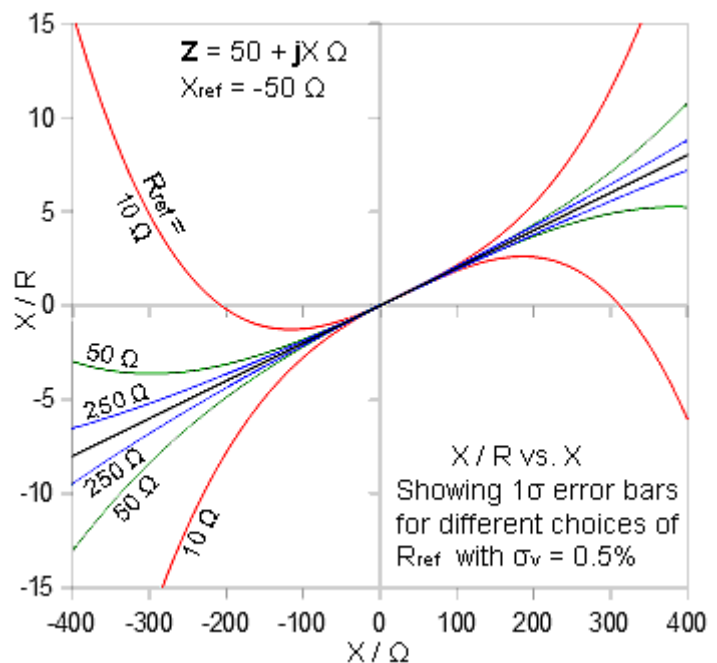
```

7c. Phase tangent error simulations

A simulation worksheet for X/R is given in the spreadsheet file [YZ_scalar_sim.ods](#) (sheet 4). Once again, if the uncertainty in the value of X_{ref} used in the explicit reference error simulation is set to the value that can be obtained from the voltage measurements (as outlined in [section 4](#)), then there is a small numerical difference between the error curves for the explicit and implicit reference methods. It is however, not large enough to warrant the extra complication of using the explicit method unless an exceptionally good reference reactance model is available; and it is not large enough to be resolved in the graph shown below.

As is generally true, a simulation is most useful when it is performed with some particular application in mind. This is definitely the case here, where a little experimentation with the numbers will show that the measurement error is fairly large when the reactive content of the impedance under test is large. As the reactance approaches zero however, the error diminishes. Thus the method does not appear to show promise for general purpose Q measurement, but it does meet the requirements of a good zero-indicating phaseometer.

The graph shown right is for an impedance $Z = 50 + jX \Omega$, with X/R plotted against X as the latter is varied between -400Ω and $+400 \Omega$. X/R therefore varies between -8 and $+8$, and is shown with error limits, i.e., $(X/R)+\sigma$ and $(X/R)-\sigma$, which are calculated from the error function for different values of R_{ref} . It will be seen that the error can be severe when R_{ref} is much smaller than the magnitude of the reactance, but as a zero indicator, the system is good over a wide range when R_{ref} is reasonably large. There is also a conjugate reactance effect, in that the minimum error is displaced away from zero to the region where X_{ref} has the opposite sign to X. The error at zero for this simulation, incidentally, is ± 0.012 (SD). The choice of X_{ref} does not have a strong effect, but results are most accurate when it is similar in magnitude to R.



Note that for a quantity that passes through zero, such as X/R, it is more useful to give an error simulation as a set of limits than as a percentage error. This is because, when $X/R = 0$, the error is infinite, even though it is small in absolute terms.

7d. Incremental error function

It is always a good idea to check a calculation procedure by performing it in two different ways to see if the results agree. This is particularly the case with error functions, because we don't usually estimate measurement uncertainty directly (by statistical sampling for example) and so the only physical test of the outcome is plausibility. Here we have generally checked a spreadsheet analysis against a Basic program. The risk with that approach however lies in the fact that both methods rely on the same mathematical derivation. Hence, if there is a mistake in the derivation that does not result in obviously-implausible output, there is a chance that it will go undetected.

In the case of error functions; the risk that subtle mistakes will occur is small for simple derivations, but it increases dramatically when repeated application of compound differentiation methods is required. Such is the case with the error functions for Q given previously.

There is however, a way in which a fair estimate of uncertainty can be made without using calculus. This is the incremental method, which involves calculating the quantity to be determined with each input measurement in turn incremented and then decremented by a small amount.

Say we determine a quantity q from a set of measurements (v_1, v_2, v_3, \dots). To do that we must apply a formula:

$$q = f(v_1, v_2, v_3, \dots)$$

Now let us increment and decrement v_1 by an amount Δv_1 and see what effect that has on q in each instance.

$$f(v_1 + \Delta v_1, v_2, v_3, \dots) = q + \Delta q_{v1+}$$

$$f(v_1 - \Delta v_1, v_2, v_3, \dots) = q - \Delta q_{v1-}$$

The reason why we perform both the positive and negative increment operations is that the magnitude of Δq will only be the same in both instances if the gradient of q vs. v_1 is a straight line in the region of that particular point in the parameter space (or if the increment is infinitesimally small). Generally the gradient will be a curve, but we can get a fairly good idea of its value at the exact point of the measurement if we take the average of the two changes. Hence:

$$\begin{aligned} \Delta q_{v1} &= \frac{(q + \Delta q_{v1+}) - (q - \Delta q_{v1-})}{2} \\ &= \frac{\Delta q_{v1+} + \Delta q_{v1-}}{2} \end{aligned}$$

So the quantity Δq_{v1} is obtained by subtracting the upper and lower calculation results and dividing by 2. From this we can get a numerical estimate of the derivative (and it will also have the same sign as the derivative if the decremented result is subtracted from the incremented result, as above). Thus:

$$\frac{\partial q}{\partial v_1} \approx \frac{\Delta q_{v1}}{\Delta v_1}$$

and using the standard deviation of v_1 , we can get an estimate of the error contribution:

$$\delta q_{v_1} = \sigma_{v_1} \frac{\partial q}{\partial v_1} \approx \sigma_{v_1} \frac{\Delta q_{v_1}}{\Delta v_1}$$

A simplification occurs however if, instead of using an arbitrary increment, we make the increment equal to the measurement standard deviation (i.e., $\Delta v_1 = \sigma_{v_1}$). Then:

$$\delta q_{v_1} = \sigma_{v_1} \frac{\partial q}{\partial v_1} \approx \Delta q_{v_1}$$

By repeating the increment and decrement operation (by an amount equal to the standard deviation of the input value) for each input value in turn, we obtain estimates for all of the error contributions. The error contributions can then be combined into an error function:

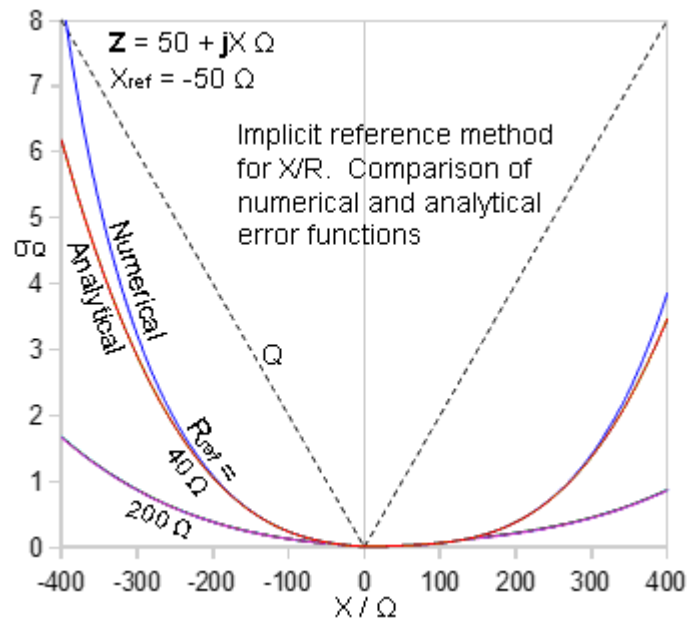
$$\sigma_q \approx \sqrt{\Delta q_{v_1}^2 + \Delta q_{v_2}^2 + \Delta q_{v_3}^2 + \dots \dots \dots}$$

This method can give very accurate results in regions where the gradient of q with respect to each of its input values is changing relatively slowly, and provided that the standard deviations are reasonably small. In that case, it will agree closely with the analytical error function, and being numerical in origin (and relatively simple to obtain) it will give an independent check on the analytical error function.

7e. Numerical error function for implicit reference X/R

The error functions for the explicit and implicit reference methods for phase tangent give a check on each other, because they produce very similar (although not exactly identical) results when the uncertainty of the reference reactance used in the explicit reference calculation is set to that which applies when that reactance is obtained from the measured voltages (section 4). The way in which they were derived in the preceding sections however means that they are somewhat interdependent in terms of provenance, for which reason an additional check is of value. There is little to be gained by checking both methods, and so an approximate numerical calculation procedure for the implicit reference case is given on the next page.

A graph comparing the analytical and numerical error calculations is shown on the right. When the error is moderate (the curve for $R_{ref} = 200 \Omega$), the two methods give almost identical results, but when the error is severe ($R_{ref} = 40 \Omega$), the numerical method is pessimistic. Note however, that we would not usually be interested in techniques that give errors comparable to the magnitude of the measurement (the dotted line shows $Q = |X|/R$), and so the region in which the numerical method breaks down is probably of little practical interest.



The Basic calculation routine listed below makes use of the existing function 'SVi_Tanphi()'. This was given in the preceding section, but it is repeated here below the calling program to facilitate cross referencing. This function can be used to calculate the quantity q by calling it with the Sign argument set to 1. Determining the incremental error function is thus turned into the trivial matter of calling the function with each of the parameters in turn incremented and decremented by its absolute standard deviation. The only additional housekeeping required is a set of checks to ensure that no divide-by-zero errors will occur. The reason for the checks is that, although the Tan ϕ function carries out its own tests, it returns zero for a divide-by-zero error, whereas an error function should return a vary large number.

Note that the use of repeated function calls in this manner is not efficient, but it makes the algorithm easy to code. Also note that when calling a function from a program in OO Basic, the arguments are separated by commas, but when calling from a spreadsheet, the arguments are separated by semicolons.

```

Function SVn_sigmaQi(ByVal sigmaV as double, offsetV as double, _
Vxz as double, Vz as double, Vx as double, Vs as double, VR as double) as double
'Calculates sigmaQ numerically according to the implicit reference method.
'Calls the function SVi_Tanphi
Dim DVxz as double, DVz as double, DVx as double, DVs as double, DVR as double
'Absolute SDs are used for the parameter displacements
DVxz = Vxz*sigmaV/100 + offsetV
DVz = Vz*sigmaV/100 + offsetV
DVx = Vx*sigmaV/100 + offsetV
DVs = Vs*sigmaV/100 + offsetV
DVR = VR*sigmaV/100 + offsetV
'Set the function value to 10^12 then check that no /0 errors will occur.
SVn_sigmaQi = 1E12
If Vs^2 - (Vxz+DVxz)^2 - VR^2 = 0 then exit function
If Vs^2 - (Vxz-DVxz)^2 - VR^2 = 0 then exit function
if (Vs+DVx)^2 - Vxz^2 - VR^2 = 0 then exit function
if (Vs-DVx)^2 - Vxz^2 - VR^2 = 0 then exit function
if Vs^2 - Vxz^2 - (VR+DVR)^2 = 0 then exit function
if Vs^2 - Vxz^2 - (VR-DVR)^2 = 0 then exit function
if Vx+DVx = 0 then exit function
if Vx-DVx = 0 then exit function
'Calculate the error contributions using the function SVi_Tanphi( )
Dim DqVxz as double, DqVz as double, DqVx as double, DqVs as double, DqVR as double
DqVxz = (SVi_Tanphi(1, Vxz+DVxz, Vz, Vx, Vs, VR)-SVi_Tanphi(1, Vxz-DVxz, Vz, Vx, Vs, VR))/2
DqVz = (SVi_Tanphi(1, Vxz, Vz+DVz, Vx, Vs, VR)-SVi_Tanphi(1, Vxz, Vz-DVz, Vx, Vs, VR))/2
DqVx = (SVi_Tanphi(1, Vxz, Vz, Vx+DVx, Vs, VR)-SVi_Tanphi(1, Vxz, Vz, Vx-DVx, Vs, VR))/2
DqVs = (SVi_Tanphi(1, Vxz, Vz, Vx, Vs+DVs, VR)-SVi_Tanphi(1, Vxz, Vz, Vx, Vs-DVs, VR))/2
DqVR = (SVi_Tanphi(1, Vxz, Vz, Vx, Vs, VR+DVR)-SVi_Tanphi(1, Vxz, Vz, Vx, Vs, VR-DVR))/2
SVn_sigmaQi = sqr( DqVxz^2 +DqVz^2 +DqVx^2 +DqVs^2 +DqVR^2 )
end function

```

```

Function SVi_Tanphi(ByVal signx as integer, Vxz as double, Vz as double, Vx as double, _
Vs as double, VR as double) as double
'Calculates Tan(phi) = X/R by the implicit reference method.
Dim w as double, Uq as double
w = Vs^2 - Vxz^2 - VR^2
if Vx = 0 or w = 0 then
SVi_Tanphi = 0
else
Uq = (Vxz^2 - Vz^2 - Vx^2)/w
SVi_Tanphi = signx*Uq*VR/Vx
end if
end function

```

8. Conductance

Admittance \mathbf{Y} is defined as the reciprocal of impedance, i.e.;

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{1}{R + jX}$$

Multiplying numerator and denominator by the complex conjugate of the denominator puts this expression into $a+jb$ form thus:

$$\mathbf{Y} = \frac{R - jX}{R^2 + X^2}$$

Which can be written¹⁰:

$$\mathbf{Y} = G + jB$$

Where the real part G is the conductance, and the numerical component B of the imaginary part is the susceptance. Hence:

$$G = \frac{R}{R^2 + X^2} \quad \text{and} \quad B = \frac{-X}{R^2 + X^2}$$

As is explained in more detail in the author's articles on impedance matching¹¹, when a pure reactance is placed in parallel with an impedance, the conductance of the combination is the same as for the original impedance. Thus, if the conductance is first adjusted so that it is equal to the reciprocal of the target load resistance, an impedance matching operation can be completed by the addition or subtraction of a parallel reactance (i.e., by adjusting the parallel element of an L-network).

By examining the analogues given in [tables 1 & 2](#), it can be seen that the combination required for conductance determination is [(1) - (2) - (4)] / (3) , i.e.;

$$\frac{|V_s|^2 - |V_{xz}|^2 - |V_R|^2}{|V_z|^2} = \frac{2R R_{ref}}{(R^2 + X^2)} = 2G R_{ref}$$

Thus, after rearrangement:

$$G = \frac{1}{2 R_{ref}} \left[\frac{|V_s|^2 - |V_{xz}|^2 - |V_R|^2}{|V_z|^2} \right] \quad [\text{Siemens}] \quad (8.1)$$

The unit ' Siemens ' (which has capital S as its symbol) is the SI name for Ω^{-1} (formerly ' Mho ').

¹⁰ Some publications give: $\mathbf{Y} = G - jB$. This can be shown to be wrong by deriving the relationship $Q = (\text{resonant frequency} / \text{bandwidth})$ for a parallel resonator. Using the wrong admittance definition gives Q as negative.

¹¹ www.g3ynh.info/zdocs/z_matching/

The error function and derivatives for conductance determination can be obtained by inspection and are given in the table below.

$G = \frac{U_G}{2R_{\text{ref}}}$ <p>where $U_G = \frac{w}{ V_Z ^2}$ and $w = V_S ^2 - V_{XZ} ^2 - V_R ^2$</p>	(8.1a)
$\sigma_G = \sqrt{\left(\sigma_{R_{\text{ref}}} \frac{\partial G}{\partial R_{\text{ref}}}\right)^2 + \left(\sigma_{V_S} \frac{\partial G}{\partial V_S }\right)^2 + \left(\sigma_{V_{XZ}} \frac{\partial G}{\partial V_{XZ} }\right)^2 + \left(\sigma_{V_R} \frac{\partial G}{\partial V_R }\right)^2 + \left(\sigma_{V_Z} \frac{\partial G}{\partial V_Z }\right)^2}$ <p>where</p> $\frac{\partial G}{\partial R_{\text{ref}}} = \frac{-U_G}{2R_{\text{ref}}^2}$ $\frac{\partial G}{\partial V_S } = \frac{ V_S }{ V_Z ^2 R_{\text{ref}}}$ $\frac{\partial G}{\partial V_{XZ} } = \frac{- V_{XZ} }{ V_Z ^2 R_{\text{ref}}}$ $\frac{\partial G}{\partial V_R } = \frac{- V_R }{ V_Z ^2 R_{\text{ref}}}$ $\frac{\partial G}{\partial V_Z } = \frac{-w}{ V_Z ^3 R_{\text{ref}}}$	(8.1u)

Conductance measurement by the scalar voltage method does not require a reference reactance (i.e., X_{ref} can be set to zero), although the choice of X_{ref} does affect the accuracy of the result. The possibility of eliminating X_{ref} , of course, also applies to the previously discussed resistance and impedance magnitude measurements ([sections 2](#) and [6](#)), but the difference in this case is that one of the measured voltages becomes redundant. Specifically:

If $X_{\text{ref}} = 0$ then $V_{XZ} = V_Z$

Thus a stand-alone conductance meter can be made by placing only a reference resistance in series with the device under test. If that is done however, then a question of statistical validity arises, and the error analysis given above might need to be altered.

The issue is that if a program routine that performs the error analysis is passed exactly the same value for both $|V_{XZ}|$ and $|V_Z|$, and the reason for that is not simply coincidence but because only one measurement exists, then the two values are correlated. In that case, the calculated uncertainty will be incorrect. We therefore need a separate error analysis for that case, but since only the derivatives involving $|V_{XZ}|$ and $|V_Z|$ will be affected, the contingency is not difficult to include.

If $|\mathbf{V}_{xz}|$ is not used, then $\frac{\partial G}{\partial |\mathbf{V}_{xz}|} = 0$

The corresponding error contribution to σ_G in the final summation is thus wiped out. The error contribution due to uncertainty in $|\mathbf{V}_z|$ however is increased, as we will see when we determine the partial derivative. We can start by writing equation (8.1) with $|\mathbf{V}_z|$ substituted in place of $|\mathbf{V}_{xz}|$.

$$G = \frac{1}{2 R_{\text{ref}}} \left[\frac{|\mathbf{V}_s|^2 - |\mathbf{V}_z|^2 - |\mathbf{V}_R|^2}{|\mathbf{V}_z|^2} \right]$$

This can be rearranged:

$$G = \frac{1}{2 R_{\text{ref}}} \left[\frac{|\mathbf{V}_s|^2 - |\mathbf{V}_R|^2}{|\mathbf{V}_z|^2} - 1 \right]$$

and so the derivative with only $|\mathbf{V}_z|$ allowed to vary is:

$$\frac{\partial G}{\partial |\mathbf{V}_z|} = \frac{-\left(|\mathbf{V}_s|^2 - |\mathbf{V}_R|^2\right)}{|\mathbf{V}_z|^3 R_{\text{ref}}}$$

Comparing this with the derivative given in the formula (8.1u), we can see that the effect of making $|\mathbf{V}_{xz}|$ redundant has been to change the magnitude of the numerator of this derivative

from $|\mathbf{V}_s|^2 - |\mathbf{V}_{xz}|^2 - |\mathbf{V}_R|^2$ to $|\mathbf{V}_s|^2 - |\mathbf{V}_R|^2$

The new numerator is of greater magnitude than the old one. The increase is also sufficient to overcome the advantage gained by losing the error contribution from $\sigma_{V_{xz}}$. The reason for that is that the redundant measurement is corroborative; i.e., in the presence of random experimental noise, combining two measurements of the same quantity reduces the uncertainty in that quantity.

Basic macro routines for G are given on the next page. Note that at the end of the argument list for the error function there is a flag called 'noX' (no reactance). If noX is zero, then it is assumed that all four voltage measurements are made using separate detectors; in which case, if $|\mathbf{V}_{xz}| = |\mathbf{V}_z|$, it is simply a coincidence resulting from the use of a very small reference reactance (the inductance of a shorting link perhaps). If $\text{noX} = 1$ and $|\mathbf{V}_{xz}| = |\mathbf{V}_z|$ however, then it is assumed that they are both the same measurement, and the derivative calculation is altered accordingly.

```

Function Scalarvolt_G(ByVal Rref as double, Vs as double, Vxz as double, VR as double, Vz as double) as double
'Calculates the conductance G of an admittance using the scalar voltage method.
'Return value is in milli Siemens (mS)
Scalarvolt_G = 1E12
if Vz = 0 then exit function
Scalarvolt_G = 1000*( (Vs^2 - Vxz^2 -VR^2)/ Vz^2)/(2*Rref)
end function

```

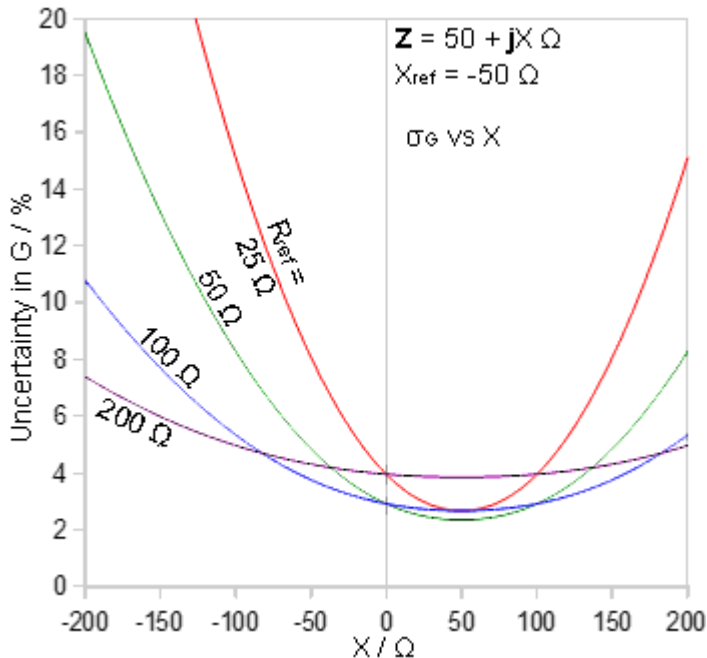
```

Function SV_sigmaG(ByVal Rref as double, sigmaRref as double, sigmaV as double, offsetV as double, Vs as double, Vxz as double, VR as double, Vz as double, noX as integer) as double
'Calculates standard deviation of G in milli Siemens (mS).
'Uncertainties are input in % and converted to absolute.
'offsetV is the voltage measurement offset (or zeroing) error, in volts.
'If noX = 1, and Vxz = Vz, assumes that Xref=0 and Vz has been input twice.
SV_sigmaG = 1E12
If Vz = 0 then exit function
Dim derRref as double, w as double
Dim sigmaVs as double, sigmaVxz as double, sigmaVR as double, sigmaVz as double
Dim derVs as double, derVxz as double, derVR as double, derVz as double
'calculate absolute uncertainties:
sigmaRref = Rref*sigmaRref/100
sigmaVs = Vs*sigmaV/100 + offsetV
sigmaVxz = Vxz*sigmaV/100 + offsetV
sigmaVR = VR*sigmaV/100 + offsetV
sigmaVz = Vz*sigmaV/100 + offsetV
'Calculate derivatives
w = Vs^2 - Vxz^2 - VR^2
derRref = -0.5 * w / (Vz * Rref)^2
derVs = Vs / (Vz^2 * Rref)
derVR = -VR / (Vz^2 * Rref)
if Vxz = Vz and noX = 1 then
  derVxz = 0
  derVz = -(Vs^2 - VR^2) / (Vz^3 * Rref)
else
  derVxz = -Vxz / (Vz^2 * Rref)
  derVz = -w / (Vz^3 * Rref)
endif
SV_sigmaG = 1000*sqr( (sigmaRref*derRref)^2 + (sigmaVs*derVs)^2 _
+(sigmaVxz*derVxz)^2 +(sigmaVR*derVR)^2 +(sigmaVz*derVz)^2 )
end function

```

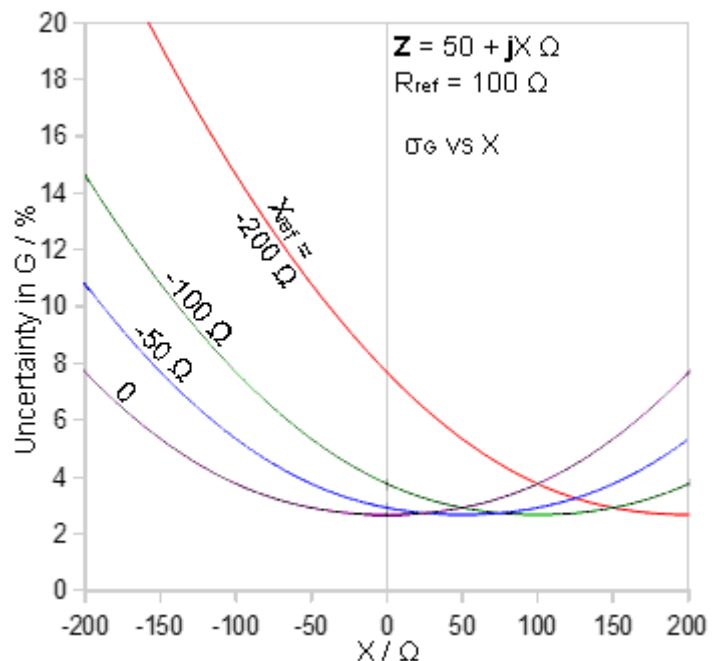
8a. Conductance error simulation

A simulation worksheet for G is given in the spreadsheet file [YZ_scalar_sim.ods](#) (sheet 5). The simulations shown in the graphs below have been carried out with a view to appraising the utility of the method in impedance-matching applications. As before, the reference resistance is taken to have a SD of 0.1%, and the voltage SD is 0.5%



This graph shows a family of curves for σ_G vs. load reactance (X) for different choices of R_{ref} . With X_{ref} set to -50Ω , and R at 50Ω , it is evident that there is a conjugate reactance effect from the positions of the error minima. The minimum achievable error occurs when $R_{ref} = R$, but the minimum is somewhat narrow, and so best accuracy over a range of load reactances occurs when R_{ref} is considerably larger than R .

This graph shows a family of curves for different reference reactances, with R_{ref} set to 100Ω . The conjugate reactance effect is evident from the way in which the error minimum always occurs when $X_{ref} = -X$. If the sign of the reference reactance is changed, the graph is flipped about $X = 0$. For a zero-indicating measuring device to be used for impedance matching, the best choice of reference reactance is obviously zero, but if the scalar voltage measurements are to be used simultaneously for the determination of X/R (phase tangent), the accuracy is still acceptable when $|X_{ref}|$ is around 50Ω .



9. Susceptance

Susceptance is the numerical component of the imaginary part of admittance. As discussed in the preceding section, it is defined as:

$$B = \frac{-X}{R^2 + X^2}$$

By examining **tables 1 & 2**, it can be seen to be obtainable from the combination [**(2)** - **(3)** - **(5)**] / **(3)**, i.e.;

$$\frac{|\mathbf{V}_{xz}|^2 - |\mathbf{V}_z|^2 - |\mathbf{V}_x|^2}{|\mathbf{V}_z|^2} = \frac{2X X_{\text{ref}}}{(R^2 + X^2)} = -2B X_{\text{ref}}$$

Thus:

$$B = \frac{-1}{2 X_{\text{ref}}} \left[\frac{|\mathbf{V}_{xz}|^2 - |\mathbf{V}_z|^2 - |\mathbf{V}_x|^2}{|\mathbf{V}_z|^2} \right] \quad [\text{Siemens}] \quad (9.1)$$

which can be simplified if so preferred:

$$B = \frac{-1}{2 X_{\text{ref}}} \left[\frac{|\mathbf{V}_{xz}|^2 - |\mathbf{V}_x|^2}{|\mathbf{V}_z|^2} - 1 \right] \quad \text{3-V method} \quad (9.1a)$$

The derivatives and error function can be obtained by inspection, and so the whole formula can be written out directly;

$B = \frac{-U_B}{2 X_{\text{ref}}} \quad \text{where} \quad U_B = \frac{ \mathbf{V}_{xz} ^2 - \mathbf{V}_z ^2 - \mathbf{V}_x ^2}{ \mathbf{V}_z ^2}$	(9.1b) 3-V method
$\sigma_{B3V} = \sqrt{\left(\sigma_{X_{\text{ref}}} \frac{\partial B}{\partial X_{\text{ref}}} \right)^2 + \left(\sigma_{V_{xz}} \frac{\partial B}{\partial \mathbf{V}_{xz} } \right)^2 + \left(\sigma_{V_z} \frac{\partial B}{\partial \mathbf{V}_z } \right)^2 + \left(\sigma_{V_x} \frac{\partial B}{\partial \mathbf{V}_x } \right)^2}$ $\frac{\partial B}{\partial X_{\text{ref}}} = \frac{U_B}{2 X_{\text{ref}}^2}$ $\frac{\partial B}{\partial \mathbf{V}_{xz} } = \frac{- \mathbf{V}_{xz} }{ \mathbf{V}_z ^2 X_{\text{ref}}}$ $\frac{\partial B}{\partial \mathbf{V}_z } = \frac{ \mathbf{V}_{xz} ^2 - \mathbf{V}_x ^2}{ \mathbf{V}_z ^3 X_{\text{ref}}}$ $\frac{\partial B}{\partial \mathbf{V}_x } = \frac{ \mathbf{V}_x }{ \mathbf{V}_z ^2 X_{\text{ref}}}$	(9.1u)

Basic macros that perform the B 3-V calculations are given below.

```

Function Scalarvolt_B3V(ByVal Xref as double, Vxz as double, Vz as double, Vx as double) as double
'Calculates the susceptance B of an admittance using the 3-V method.
'Return value is in milli Siemens (mS)
Scalarvolt_B3V = 1E12
if Vz = 0 then exit function
if Xref = 0 then exit function
Scalarvolt_B3V = -1000*(Vxz^2 - Vz^2 -Vx^2)/(2*Vz^2*Xref)
end function

Function SV_sigmaB3V(ByVal Xref as double, sigmaXref as double, sigmaV as double, offsetV as double, Vxz as double, Vz as double, Vx as double) as double
'Calculates sigmaB using the 3-V method.
'standard deviations are input as percentages and converted into absolute SD
'offsetV is the voltage measurement offset (or zeroing) error, in volts.
'Return value is in milli Siemens (mS)
SV_sigmaB3V = 1E12
if Xref = 0 then exit function
if Vz = 0 then exit function
SigmaXref = Xref*sigmaXref/100
Dim sigmaVxz as double, sigmaVz as double, sigmaVx as double
sigmaVxz = Vxz*sigmaV/100 + offsetV
sigmaVz = Vz*sigmaV/100 + offsetV
sigmaVx = Vx*sigmaV/100 + offsetV
Dim derXref as double, derVxz as double, derVz as double, derVx as double
derXref = (Vxz^2-Vz^2-Vx^2)/(2*Vz^2*Xref^2)
derVxz = -Vxz/(Vz^2*Xref)
derVz = (Vxz^2-Vx^2)/(Vz^3*Xref)
derVx = Vx/(Vz^2*Xref)
SV_sigmaB3V = 1000*sqr( (sigmaXref*derXref)^2 +(sigmaVxz*derVxz)^2 _
+(sigmaVz*derVz)^2 +(sigmaVx*derVx)^2 )
end function

```

The 3-V susceptance calculation, of course, requires explicit knowledge of the reference reactance. As has been established in previous sections, it is generally easier to obtain the reference reactance implicitly from the reference resistance, and so we can use the substitution given earlier as equation (4.1) in section 4.

$$X_{\text{ref}} = \text{Sign}(X_{\text{ref}}) R_{\text{ref}} \frac{|V_X|}{|V_R|}$$

Substituting this into (9.1) we have:

$$B = \frac{-\text{Sign}(X_{\text{ref}}) |V_R|}{2 R_{\text{ref}} |V_X|} \left[\frac{|V_{XZ}|^2 - |V_Z|^2 - |V_X|^2}{|V_Z|^2} \right] \quad [\text{Siemens}] \quad (9.2)$$

or

$$B = \frac{-\text{Sign}(X_{\text{ref}}) |V_R|}{2 R_{\text{ref}} |V_X|} \left[\frac{|V_{XZ}|^2 - |V_X|^2}{|V_Z|^2} - 1 \right] \quad \text{4-V method} \quad (9.2a)$$

This requires four voltages, and so will be referred to as the 4-V method for B.

The derivatives for the error function are, once again, trivial; although $\partial B / \partial |V_X|$ is easier to obtain if (9.2) is rearranged:

$$B = \frac{-\text{Sign}(X_{\text{ref}}) |V_R|}{2 R_{\text{ref}} |V_Z|^2} \left[\frac{|V_{XZ}|^2 - |V_Z|^2}{|V_X|} - |V_X| \right]$$

hence, differentiating the two terms with respect to $|V_X|$ gives:

$$\frac{\partial B}{\partial |V_X|} = \frac{-\text{Sign}(X_{\text{ref}}) |V_R|}{2 R_{\text{ref}} |V_Z|^2} \left[\frac{-|V_{XZ}|^2 + |V_Z|^2}{|V_X|^2} - 1 \right] \quad [\text{Siemens / volt}]$$

and multiplying the leading algebraic sign into the bracket gives:

$$\frac{\partial B}{\partial |V_X|} = \frac{\text{Sign}(X_{\text{ref}}) |V_R|}{2 R_{\text{ref}} |V_Z|^2} \left[\frac{|V_{XZ}|^2 - |V_Z|^2}{|V_X|^2} + 1 \right]$$

As before, the Sign operator is carried over into all of the derivatives; but since a standard deviation is a magnitude, we do not need to use the sign when evaluating the error function.

The formula for susceptance using the 4-V method is given on the next page. Note that the following equivalence has been used to de-clutter some of the equations:

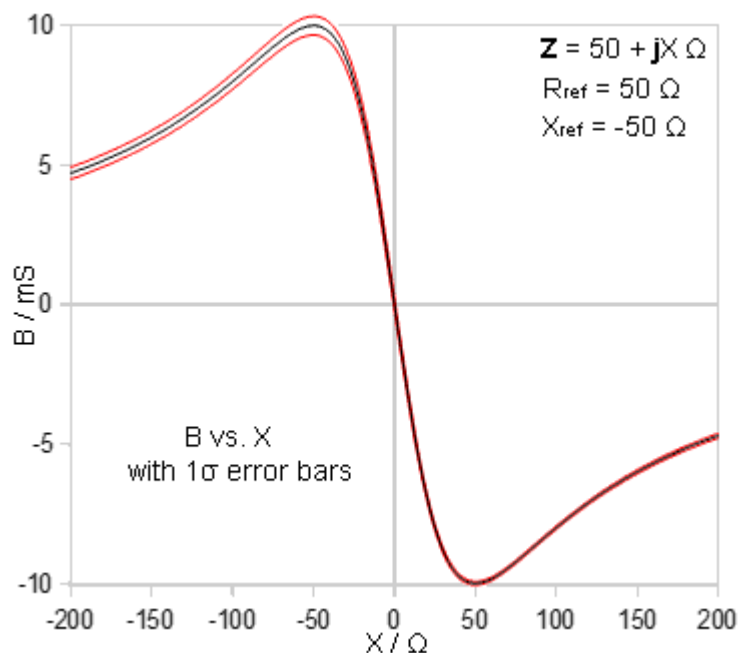
$$\frac{|V_R|}{|V_X|} \equiv \frac{|V_R|}{|V_X|}$$

$B = -\text{Sign}(X_{\text{ref}}) \frac{ \mathbf{V}_R }{ \mathbf{V}_X } \frac{U_B}{2R_{\text{ref}}} \quad \text{where} \quad U_B = \frac{ \mathbf{V}_{XZ} ^2 - \mathbf{V}_Z ^2 - \mathbf{V}_X ^2}{ \mathbf{V}_Z ^2}$	(9.2b) 4-V method
$\sigma_{B4V} = \sqrt{\left(\sigma_{R_{\text{ref}}} \frac{\partial B}{\partial R_{\text{ref}}}\right)^2 + \left(\sigma_{V_R} \frac{\partial B}{\partial \mathbf{V}_R }\right)^2 + \left(\sigma_{V_{XZ}} \frac{\partial B}{\partial \mathbf{V}_{XZ} }\right)^2 + \left(\sigma_{V_Z} \frac{\partial B}{\partial \mathbf{V}_Z }\right)^2 + \left(\sigma_{V_X} \frac{\partial B}{\partial \mathbf{V}_X }\right)^2}$ $\frac{\partial B}{\partial R_{\text{ref}}} = \text{Sign}(X_{\text{ref}}) \frac{ \mathbf{V}_R }{ \mathbf{V}_X } \frac{U_B}{2R_{\text{ref}}^2}$ $\frac{\partial B}{\partial \mathbf{V}_R } = -\text{Sign}(X_{\text{ref}}) \frac{1}{ \mathbf{V}_X } \frac{U_B}{2R_{\text{ref}}}$ $\frac{\partial B}{\partial \mathbf{V}_{XZ} } = -\text{Sign}(X_{\text{ref}}) \frac{ \mathbf{V}_R }{ \mathbf{V}_X } \frac{ \mathbf{V}_{XZ} }{ \mathbf{V}_Z ^2 R_{\text{ref}}}$ $\frac{\partial B}{\partial \mathbf{V}_Z } = \text{Sign}(X_{\text{ref}}) \frac{ \mathbf{V}_R }{ \mathbf{V}_X } \frac{(\mathbf{V}_{XZ} ^2 - \mathbf{V}_X ^2)}{ \mathbf{V}_Z ^3 R_{\text{ref}}}$ $\frac{\partial B}{\partial \mathbf{V}_X } = \text{Sign}(X_{\text{ref}}) \frac{ \mathbf{V}_R }{2 \mathbf{V}_Z ^2 R_{\text{ref}}} \left(1 + \frac{ \mathbf{V}_{XZ} ^2 - \mathbf{V}_Z ^2}{ \mathbf{V}_X ^2}\right)$	(9.2u)

An error simulation worksheet for B, comparing the 3-V and 4-V methods, is given in the spreadsheet file [YZ_scalar_sim.ods](#) (sheet 6). If the uncertainty in X_{ref} for the 3-V method is set to the value that is obtained by determining X_{ref} from the voltage measurements, then the two methods give very similar results.

The graph on the right shows B vs. X with 1σ error limits calculated using the 4-V method ($\sigma_{R_{\text{ref}}} = 0.1\%$, $\sigma_V = 0.5\%$). The error in the region of zero is ± 0.25 mS. Minimum error occurs when X_{ref} has the same magnitude as X but the opposite sign (conjugate reactance effect).

When straight percentages are used for the input errors, the choice of X_{ref} and R_{ref} makes no difference to the simulation.



Basic macros that perform the B 4-V calculations are given below.

```

Function Scalarvolt_B4V(ByVal signX as integer, Rref as double, VR as double, Vxz as double, Vz as double, Vx as double) as double
'Calculates the susceptance B of an admittance using the 4-V method.
'signX is the sign of the reference reactance (-1 for a capacitor used below its SRF)
'Return value is in milli Siemens (mS)
Scalarvolt_B4V = 1E12
if Vz = 0 then exit function
if Vx = 0 then exit function
if Rref = 0 then exit function
Scalarvolt_B4V = -1000*signX*VR*(Vxz^2 - Vz^2 - Vx^2)/(2*Vx*Vz^2*Rref)
end function

Function SV_sigmaB4V(ByVal Rref as double, sigmaRref as double, sigmaV as double, offsetV as double, VR as double, Vxz as double, Vz as double, Vx as double) as double
'Calculates sigmaB using the 4-V method.
'standard deviations are input as percentages and converted into absolute SD
'offsetV is the voltage measurement offset (or zeroing) error, in volts.
'Return value is in milli Siemens (mS)
SV_sigmaB4V = 1E12
if Rref = 0 then exit function
if Vx = 0 then exit function
if Vz = 0 then exit function
SigmaRref = Rref*sigmaRref/100
Dim sigmaVR as double, sigmaVxz as double, sigmaVz as double, sigmaVx as double
sigmaVR = VR*sigmaV/100 + offsetV
sigmaVxz = Vxz*sigmaV/100 + offsetV
sigmaVz = Vz*sigmaV/100 + offsetV
sigmaVx = Vx*sigmaV/100 + offsetV
Dim UB as double, derRref as double, derVR as double
Dim derVxz as double, derVz as double, derVx as double
UB = (Vxz^2-Vz^2-Vx^2)/Vz^2
derRref = (VR/Vx)*UB/(2*Rref^2)
derVR = -UB/(2*Vx*Rref)
derVxz = -Vxz*(VR/Vx)/(Vz^2*Rref)
derVz = (VR/Vx)*(Vxz^2-Vx^2)/(Vz^3*Rref)
derVx = VR*(1+(Vxz^2-Vz^2)/Vx^2)/(2*Vz^2*Rref)
SV_sigmaB4V = 1000*sqr( (sigmaRref*derRref)^2 +(sigmaVR*derVR)^2 _
+(sigmaVxz*derVxz)^2 +(sigmaVz*derVz)^2 +(sigmaVx*derVx)^2 )
end function

```


10. Power factor

The power factor of an impedance is the cosine of the phase angle. Hence :

$$\text{PF} = \cos \phi = \frac{R}{|Z|} = G|Z| = \frac{R}{\sqrt{R^2 + X^2}} \quad [\text{Dimensionless}]$$

Using the analogues in [tables 1 and 2](#), we can obtain it using [(1) - (2) - (4)] / [$\sqrt{(3)} \times \sqrt{(4)}$], i.e.:

$$\frac{|V_S|^2 - |V_{XZ}|^2 - |V_R|^2}{|V_Z| |V_R|} = \frac{2R}{\sqrt{R^2 + X^2}}$$

Hence:

$$\text{PF} = \frac{1}{2} \left(\frac{|V_S|^2 - |V_{XZ}|^2 - |V_R|^2}{|V_Z| |V_R|} \right) \quad (10.1)$$

From this expression it can be seen that, for the determination of PF, a finite voltage is required across R_{ref} , but it is not necessary to know the actual resistance. A voltage across X_{ref} however, is not required, and so X_{ref} can be set to zero if so desired. When the reference reactance is shorted-out however, $|V_{XZ}|$ becomes equal to $|V_Z|$, and the error function will need to be modified if the same measurement is to be used twice. We met this issue previously when deriving the error function for conductance, and it can be handled in much the same way.

In the matter of deriving the error function, the derivatives for the non-zero reference reactance case can be had by inspection; except perhaps for $\partial \text{PF} / \partial |V_R|$, which is easier to deduce if (10.1) is rearranged:

$$\text{PF} = \frac{1}{2} \left(\frac{|V_S|^2 - |V_{XZ}|^2}{|V_Z| |V_R|} - \frac{|V_R|}{|V_Z|} \right)$$

Hence:

$$\frac{\partial \text{PF}}{\partial |V_R|} = \frac{1}{2} \left(\frac{|V_S|^2 - |V_{XZ}|^2}{|V_Z| (-|V_R|^2)} - \frac{1}{|V_Z|} \right)$$

which, after factoring the (-) sign from the bracket gives:

$$\frac{\partial \text{PF}}{\partial |V_R|} = -\frac{1}{2} \left(\frac{|V_S|^2 - |V_{XZ}|^2}{|V_Z| |V_R|^2} + \frac{1}{|V_Z|} \right) \quad [\text{Volts}^{-1}]$$

If the reference reactance is zero, and the same voltage is input for both $|V_{XZ}|$ and $|V_Z|$, then the required modification to the error function can be deduced by rewriting (10.1) thus:

$$\text{PF} = \frac{1}{2} \left(\frac{|V_S|^2 - |V_Z|^2 - |V_R|^2}{|V_Z| |V_R|} \right) \quad \text{when } V_{XZ} \text{ is replaced by } V_Z$$

In this case, $\partial \text{PF} / \partial |\mathbf{V}_{\text{xz}}|$ is zero, and the new version of $\partial \text{PF} / \partial |\mathbf{V}_{\text{z}}|$ is easiest to obtain if we make the rearrangement:

$$\text{PF} = \frac{1}{2} \left(\frac{|\mathbf{V}_{\text{s}}|^2 - |\mathbf{V}_{\text{r}}|^2}{|\mathbf{V}_{\text{z}}| |\mathbf{V}_{\text{r}}|} - \frac{|\mathbf{V}_{\text{z}}|}{|\mathbf{V}_{\text{r}}|} \right)$$

Hence:

$$\frac{\partial \text{PF}}{\partial |\mathbf{V}_{\text{z}}|} = -\frac{1}{2} \left(\frac{|\mathbf{V}_{\text{s}}|^2 - |\mathbf{V}_{\text{r}}|^2}{|\mathbf{V}_{\text{r}}| |\mathbf{V}_{\text{z}}|^2} + \frac{1}{|\mathbf{V}_{\text{r}}|} \right) \quad \text{when } \mathbf{V}_{\text{xz}} \text{ is replaced by } \mathbf{V}_{\text{z}}$$

The complete set of formulae for power factor determination is as follows:

$\text{PF} = \left(\frac{ \mathbf{V}_{\text{s}} ^2 - \mathbf{V}_{\text{xz}} ^2 - \mathbf{V}_{\text{r}} ^2}{2 \mathbf{V}_{\text{z}} \mathbf{V}_{\text{r}} } \right)$	(10.1a)
$\sigma_{\text{PF}} = \sqrt{\left(\sigma_{\text{vs}} \frac{\partial \text{PF}}{\partial \mathbf{V}_{\text{s}} } \right)^2 + \left(\sigma_{\text{vxz}} \frac{\partial \text{PF}}{\partial \mathbf{V}_{\text{xz}} } \right)^2 + \left(\sigma_{\text{vr}} \frac{\partial \text{PF}}{\partial \mathbf{V}_{\text{r}} } \right)^2 + \left(\sigma_{\text{vz}} \frac{\partial \text{PF}}{\partial \mathbf{V}_{\text{z}} } \right)^2}$ <p>where</p> $\frac{\partial \text{PF}}{\partial \mathbf{V}_{\text{s}} } = \frac{ \mathbf{V}_{\text{s}} }{ \mathbf{V}_{\text{z}} \mathbf{V}_{\text{r}} }$ $\frac{\partial \text{PF}}{\partial \mathbf{V}_{\text{xz}} } = \frac{- \mathbf{V}_{\text{xz}} }{ \mathbf{V}_{\text{z}} \mathbf{V}_{\text{r}} }$ $\frac{\partial \text{PF}}{\partial \mathbf{V}_{\text{r}} } = \frac{ \mathbf{V}_{\text{xz}} ^2 - \mathbf{V}_{\text{s}} ^2}{2 \mathbf{V}_{\text{z}} \mathbf{V}_{\text{r}} ^2} - \frac{1}{2 \mathbf{V}_{\text{z}} }$ $\frac{\partial \text{PF}}{\partial \mathbf{V}_{\text{z}} } = -\frac{(\mathbf{V}_{\text{s}} ^2 - \mathbf{V}_{\text{xz}} ^2 - \mathbf{V}_{\text{r}} ^2)}{2 \mathbf{V}_{\text{r}} \mathbf{V}_{\text{z}} ^2}$ <p>unless $\mathbf{V}_{\text{xz}} \equiv \mathbf{V}_{\text{z}}$ (i.e., they are the same measurement), in which case:</p> $\frac{\partial \text{PF}}{\partial \mathbf{V}_{\text{xz}} } = 0$ $\frac{\partial \text{PF}}{\partial \mathbf{V}_{\text{z}} } = \frac{ \mathbf{V}_{\text{r}} ^2 - \mathbf{V}_{\text{s}} ^2}{2 \mathbf{V}_{\text{r}} \mathbf{V}_{\text{z}} ^2} - \frac{1}{2 \mathbf{V}_{\text{r}} }$	(10.1u)

An error simulation is given in the spreadsheet file [YZ_scalar_sim.ods](#) (sheet 5). This is the same worksheet as was used for conductance, the two quantities being closely related. For the same voltage measurement percentage uncertainty, the calculated percentage uncertainty simulation results for PF are the same as those for G (see [section 8](#)).

Basic macros for the power factor calculations are given below:

```
Function Scalarvolt_PF(ByVal Vs as double, Vxz as double, VR as double, _
Vz as double) as double
'Calculates the power factor of an impedance using the scalar voltage method.
Scalarvolt_PF = 0
if Vz = 0 then exit function
if VR = 0 then exit function
Scalarvolt_PF = (Vs^2 -Vxz^2 -VR^2)/(2*Vz*VR)
end function

Function SV_sigmaPF(ByVal sigmaV as double, offsetV as double, Vs as double, _
Vxz as double, VR as double, Vz as double, nox as integer) as double
'Calculates standard deviation of power factor
'Uncertainties are input in % and converted to absolute.
'offsetV is the voltage measurement offset (or zeroing) error, in volts.
'If nox = 1, and Vxz = Vz, assumes that Vxz and Vz are the same measurement.
SV_sigmaPF = 1
If Vz = 0 then exit function
if VR = 0 then exit function
Dim sigmaVs as double, sigmaVxz as double, sigmaVR as double, sigmaVz as double
Dim derVs as double, derVxz as double, derVR as double, derVz as double
'calculate absolute uncertainties:
sigmaVs = Vs*sigmaV/100 + offsetV
sigmaVxz = Vxz*sigmaV/100 + offsetV
sigmaVR = VR*sigmaV/100 + offsetV
sigmaVz = Vz*sigmaV/100 + offsetV
'Calculate derivatives
derVs = Vs / (Vz * VR)
derVR = -(Vs^2 -Vxz^2)/(2*Vz*VR^2) -1/(2*Vz)
if Vxz = Vz and nox = 1 then
    derVxz = 0
    derVz = -(Vs^2 -VR^2)/(2*VR*Vz^2) -1/(2*VR)
else
    derVxz = -Vxz / (Vz*VR)
    derVz = -(Vs^2-Vxz^2-VR^2)/(2*VR*Vz^2)
endif
SV_sigmaPF = sqr((sigmaVs*derVs)^2 +(sigmaVxz*derVxz)^2 _
+(sigmaVR*derVR)^2 +(sigmaVz*derVz)^2 )
end function
```

Note that power factor varies between 0 and 1, and so the maximum possible uncertainty in the event of an invalid input is 1.

11. Collected formulae

The formulae that have been derived so far are summarised in the table below. For compactness, the following substitutions are used:

$$u = |\mathbf{V}_{xz}|^2 - |\mathbf{V}_z|^2 - |\mathbf{V}_x|^2 \quad \text{and} \quad w = |\mathbf{V}_s|^2 - |\mathbf{V}_{xz}|^2 - |\mathbf{V}_R|^2$$

			section
R	$R = \frac{R_{\text{ref}}}{2} \frac{w}{ \mathbf{V}_R ^2}$		2
X	$X = \frac{X_{\text{ref}}}{2} \frac{u}{ \mathbf{V}_X ^2}$	3-V method	3
	$X = \text{sign} \frac{R_{\text{ref}}}{2} \frac{u}{ \mathbf{V}_R \mathbf{V}_X }$	4-V method	5
Z	$ \mathbf{Z} = R_{\text{ref}} \left \frac{\mathbf{V}_Z}{\mathbf{V}_R} \right $		6
Tanφ	$\frac{X}{R} = \frac{R_{\text{ref}}}{X_{\text{ref}}} \frac{u}{w}$	explicit ref.	7
	$\frac{X}{R} = \text{sign} \left \frac{\mathbf{V}_R}{\mathbf{V}_X} \right \frac{u}{w}$	implicit ref.	
PF	$\cos \phi = \frac{1}{2} \frac{w}{ \mathbf{V}_Z \mathbf{V}_R }$		10
G	$G = \frac{1}{2 R_{\text{ref}}} \frac{w}{ \mathbf{V}_Z ^2}$		8
B	$B = \frac{-1}{2 X_{\text{ref}}} \frac{u}{ \mathbf{V}_Z ^2}$	3-V method	9
	$B = -\text{sign} \frac{1}{2 R_{\text{ref}}} \left \frac{\mathbf{V}_R}{\mathbf{V}_X} \right \frac{u}{ \mathbf{V}_Z ^2}$	4-V method	

Symmetry is apparent in the relationship between R and X, and in the relationship between G and B, when the reference reactance is used explicitly; but it is less apparent when X_{ref} is eliminated.

The quantity 'sign' is the numerical representation of the sign of the reference reactance (-1 or +1).

The product or quotient of the magnitudes of two complex numbers is the same as the magnitude of the product or quotient; so:

$$|\mathbf{a}| |\mathbf{b}| = |\mathbf{a} \mathbf{b}| \quad \text{and} \quad |\mathbf{a}| / |\mathbf{b}| = |\mathbf{a} / \mathbf{b}|$$

These correspondences have been used to de-clutter the equations.

12. Reflection coefficient and SWR

The ubiquitous SWR meter or SWR bridge does not measure SWR. It actually indicates a quantity known as the *voltage reflection coefficient magnitude*, which varies between zero, for a perfectly matched load, and 1 for a completely mismatched load (open, short, or pure reactance). This can be converted into SWR if, and only if, an electrical quarter wavelength or more of uniform transmission line of known characteristic resistance is interposed between the measuring point and the load, and if the generator output impedance is equal to the characteristic resistance. SWR readings in HF radio practice are therefore mostly nonsense¹², but the practice of replacing the 0 to 1 scale of the panel-meter with a non-linear SWR scale running from 1 to ∞ persists. Nominal SWR, whether fact or gibberish, is in any case calculated from the reflection coefficient, and so we will start by defining the actual measured quantity.

Voltage reflection coefficient is the ratio of the amplitude of the wave reflected from the load to the amplitude of the wave absorbed by the load, as inferred relative to a specified characteristic resistance. A conventional in-line reflectometer measures the relative amplitudes of the forward and reverse travelling waves on a transmission line by taking voltage samples from one, or sometimes two, potential dividers (or voltage transformers) placed across the line, and current samples from a current transformer placed in series with the line. It is convenient to call the voltage from the potential divider V_v , and the voltage from the current transformer V_i . Both of these quantities are phasors, and the sampling networks are engineered to preserve the relative phase information in the transmission-line voltage and current at the measuring point. The voltages are also scaled by choice of circuit parameters so that, when the load impedance Z is the same as the line characteristic resistance R_0 , $V_v = V_i$. When these conditions are met, $V_v - V_i$ is proportional to the reflected wave amplitude, and $V_v + V_i$ is proportional to the forward wave amplitude. The current transformer also has the convenient property that it can provide outputs that float relative to ground, and so vector addition can be performed simply by placing a current sample in series with a voltage sample, series aiding for the forward voltage, and in series opposition for the reflected voltage. The ratio of the reflected and forward voltages is the voltage reflection coefficient, Γ (Gamma). When we rectify the two voltage sums however, we lose the phase information, and so we measure the quantity $|\Gamma|$ (the VRC magnitude). Thus:

$$|\Gamma| = \frac{|V_{\text{rev}}|}{|V_{\text{fwd}}|} = \frac{|V_v - V_i|}{|V_v + V_i|} \quad (12.1)$$

Note that constants of proportionality vanish by cancellation because we are dealing with ratios. Also, although it is perfectly feasible to measure these voltages and compute the ratio (and hence the SWR if so desired); the common practice in low-cost instruments is to provide a sensitivity control and two (nominally) identically calibrated panel meters, one showing $|V_v - V_i|$ and the other showing $|V_v + V_i|$. If the sensitivity is adjusted until the $|V_v + V_i|$ reading is 1, then the other meter reads the reflection coefficient on a linear scale from 0 to 1 (neglecting diode non-linearity). The VSWR¹³, incidentally, is given by:

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

and this formula is used to determine the panel-meter scale engraving of an SWR bridge.

¹² Particularly when an SWR meter is used at the input to an antenna matching unit, and ludicrously so when the SWR meter is actually built-in to the AMU.

¹³ Voltage Standing-Wave Ratio, so called to distinguish it from power SWR. Voltage and current SWR are the same.

A bridge however, is an impedance measuring device. It does not need to meet any preconditions relating to transmission lines in order to produce a valid result. It can also be used as a generalised analogue computer for impedance and admittance attributes, because it has the property that if one of the samples is defined as an analogue, the other becomes the corresponding analogue of the reference impedance. In the case of the reflection coefficient bridge, the reference impedance is R_0 , the target load resistance or transmission-line surge resistance, which has been built into the balance condition by choice of circuit parameters. In that case, V_v becomes the analogue of Z (the actual load impedance) and V_i becomes the analogue of R_0 . Substituting the analogues into (12.1) we get:

$$|\Gamma| = \frac{|Z - R_0|}{|Z + R_0|} \quad (12.2)$$

This relationship, incidentally, is well known and can be derived in numerous ways^{14 15 16}.

So, now we have an expression for $|\Gamma|$ in terms of quantities that can be measured by the discrete scalar voltage method; although, as we will soon discover, this will only work when $R_{ref} = R_0$. The possibility of using this approach however, raises the question: 'why don't antenna analysers do it that way?' We can, of course, answer that question by comparing the scalar method with the conventional bridge method.

When using the scalar voltage method, we will work with the square of the VRC magnitude. This will dispense with the proliferation of square-root symbols that occurs when dealing with magnitudes. Also, the quantity $|\Gamma|^2$ has a name; it is the *power reflection coefficient magnitude* (PRC). Thus, representing Z as $R + jX$, and squaring and expanding (12.2) we get:

$$|\Gamma|^2 = \frac{(R - R_0)^2 + X^2}{(R + R_0)^2 + X^2} \quad (12.3)$$

Now, examining tables 2 and 3, we can see that an expression in the same form as the numerator of (12.3) is obtained from: (3) + (4) - [(1) - (2) - (4)]; i.e.; (2) + (3) + 2×(4) - (1).

$$|V_{xz}|^2 + |V_z|^2 + 2|V_R|^2 - |V_s|^2 = |I|^2 (R^2 + X^2 + R_{ref}^2 - 2R R_{ref}) = |I|^2 [(R - R_{ref})^2 + X^2]$$

and an expression in the form of the denominator is obtained from: (1) - (5) - [(2) - (3) - (5)]; i.e.; (1) + (3) - (2).

$$|V_s|^2 + |V_z|^2 - |V_{xz}|^2 = |I|^2 [(R + R_{ref})^2 + X^2]$$

Hence, if we put $R_{ref} = R_0$;

$$|\Gamma|^2 = \frac{|V_{xz}|^2 + |V_z|^2 + 2|V_R|^2 - |V_s|^2}{|V_s|^2 + |V_z|^2 - |V_{xz}|^2} \quad \text{when } R_{ref} = R_0 \quad (12.4)$$

14 See, for example: **Transmission Lines for Digital and Communication Networks**, R E Matlick, McGraw-Hill 1969. LCCN: 68-30561. Section 2.9, Reflection Coefficient, p43-46.

15 A good discussion is given in: **Return Loss**, Thomas Roddam, Wireless World. Nov. 1957, p521-524, Dec. 1957, p583-588.

16 A summary is given in: **Electronic Measurements**, F E Terman and J M Petit, 2nd Edition, McGraw-Hill 1952, LCCN 51-12650. Section 4.3, Voltage and Current Relations, Reflection Coefficient and SWR. p125-128.

So far, interestingly, we are at liberty to retain a non-zero value for X_{ref} , but the expression is rendered somewhat complicated by that choice. If we put $X_{\text{ref}} = 0$, then $\mathbf{V}_{\text{XZ}} = \mathbf{V}_Z$, and a simplification occurs:

$$|\Gamma|^2 = \frac{2|\mathbf{V}_Z|^2 + 2|\mathbf{V}_R|^2 - |\mathbf{V}_S|^2}{|\mathbf{V}_S|^2} \quad \text{when } R_{\text{ref}} = R_0 \quad \text{and} \quad X_{\text{ref}} = 0 \quad (12.5)$$

or, alternatively:

$$|\Gamma|^2 = \frac{2(|\mathbf{V}_Z|^2 + |\mathbf{V}_R|^2)}{|\mathbf{V}_S|^2} - 1 \quad \text{when } R_{\text{ref}} = R_0 \quad \text{and} \quad X_{\text{ref}} = 0 \quad (12.5a)$$

PRC error function

An error function for the power reflection coefficient can be written by inspection of equation (12.4). Four voltage measurements are required when $X_{\text{ref}} \neq 0$, and so:

$$\sigma_{\text{PRC}} = \sqrt{\left(\sigma_{\text{VXZ}} \frac{\partial |\Gamma|^2}{\partial |\mathbf{V}_{\text{XZ}}|}\right)^2 + \left(\sigma_{\text{VZ}} \frac{\partial |\Gamma|^2}{\partial |\mathbf{V}_Z|}\right)^2 + \left(\sigma_{\text{VR}} \frac{\partial |\Gamma|^2}{\partial |\mathbf{V}_R|}\right)^2 + \left(\sigma_{\text{VS}} \frac{\partial |\Gamma|^2}{\partial |\mathbf{V}_S|}\right)^2}$$

To facilitate differentiation we will write (12.4) as:

$$|\Gamma|^2 = \frac{|\mathbf{V}_{\text{XZ}}|^2 + |\mathbf{V}_Z|^2 + 2|\mathbf{V}_R|^2 - |\mathbf{V}_S|^2}{|\mathbf{V}_S|^2 + |\mathbf{V}_Z|^2 - |\mathbf{V}_{\text{XZ}}|^2} = \frac{\alpha}{\beta}$$

where:

$$\alpha = |\mathbf{V}_{\text{XZ}}|^2 + |\mathbf{V}_Z|^2 + 2|\mathbf{V}_R|^2 - |\mathbf{V}_S|^2$$

and

$$\beta = |\mathbf{V}_S|^2 + |\mathbf{V}_Z|^2 - |\mathbf{V}_{\text{XZ}}|^2$$

and the following combinations will be found useful:

$$\alpha + \beta = 2(|\mathbf{V}_Z|^2 + |\mathbf{V}_R|^2)$$

$$\beta - \alpha = 2(|\mathbf{V}_S|^2 - |\mathbf{V}_{\text{XZ}}|^2 - |\mathbf{V}_R|^2) = 2w$$

(In previous sections, the letter w was allocated to the combination in brackets above).

Differentiation can be accomplished using the quotient rule, i.e.;

$$\frac{d\left(\frac{\alpha}{\beta}\right)}{dv} = \frac{\beta \frac{d\alpha}{dv} - \alpha \frac{d\beta}{dv}}{\beta^2}$$

Hence:

$$\frac{\partial |\Gamma|^2}{\partial |\mathbf{V}_{xz}|} = \frac{\beta 2|\mathbf{V}_{xz}| - \alpha(-2|\mathbf{V}_{xz}|)}{\beta^2} = \frac{2|\mathbf{V}_{xz}|(\alpha + \beta)}{\beta^2}$$

which, after substituting for $\alpha + \beta$, gives:

$$\frac{\partial |\Gamma|^2}{\partial |\mathbf{V}_{xz}|} = \frac{4|\mathbf{V}_{xz}|(|\mathbf{V}_z|^2 + |\mathbf{V}_R|^2)}{\beta^2} \quad [\text{volts}^{-1}] \quad (12.4-d1)$$

$$\frac{\partial |\Gamma|^2}{\partial |\mathbf{V}_z|} = \frac{\beta 2|\mathbf{V}_z| - \alpha 2|\mathbf{V}_z|}{\beta^2} = \frac{2|\mathbf{V}_z|(\beta - \alpha)}{\beta^2}$$

which, after replacing $\beta - \alpha$, gives:

$$\frac{\partial |\Gamma|^2}{\partial |\mathbf{V}_z|} = \frac{4|\mathbf{V}_z|(|\mathbf{V}_s|^2 - |\mathbf{V}_{xz}|^2 - |\mathbf{V}_R|^2)}{\beta^2} \quad [\text{volts}^{-1}] \quad (12.4-d2)$$

$$\frac{\partial |\Gamma|^2}{\partial |\mathbf{V}_R|} = \frac{4|\mathbf{V}_R|}{\beta} \quad [\text{volts}^{-1}] \quad (12.4-d3)$$

and

$$\frac{\partial |\Gamma|^2}{\partial |\mathbf{V}_s|} = \frac{\beta(-2|\mathbf{V}_s|) - \alpha 2|\mathbf{V}_s|}{\beta^2} = \frac{-2|\mathbf{V}_s|(\alpha + \beta)}{\beta^2}$$

which, after replacing $\alpha + \beta$, gives:

$$\frac{\partial |\Gamma|^2}{\partial |\mathbf{V}_s|} = \frac{-4|\mathbf{V}_s|(|\mathbf{V}_z|^2 + |\mathbf{V}_R|^2)}{\beta^2} \quad [\text{volts}^{-1}] \quad (12.4-d4)$$

If $X_{ref} = 0$ however, and $|\mathbf{V}_{xz}|$ and $|\mathbf{V}_z|$ are both the same measurement, then we have the somewhat more straightforward task of differentiating equation (12.5a):

$$\frac{\partial |\Gamma|^2}{\partial |\mathbf{V}_{xz}|} = 0 \quad (12.5-d1)$$

$$\frac{\partial |\Gamma|^2}{\partial |\mathbf{V}_z|} = \frac{4|\mathbf{V}_z|}{|\mathbf{V}_s|^2} \quad (12.5-d2)$$

$$\frac{\partial |\Gamma|^2}{\partial |\mathbf{V}_R|} = \frac{4|\mathbf{V}_R|}{|\mathbf{V}_s|^2} \quad (12.5-d3)$$

and

$$\frac{\partial |\Gamma|^2}{\partial |\mathbf{V}_s|} = \frac{-4(|\mathbf{V}_z|^2 + |\mathbf{V}_R|^2)}{|\mathbf{V}_s|^3} \quad (12.5-d4)$$

In the calculation routines for the standard deviations of conductance and power factor however, we used an input flag (noX) to tell the program when the same measurement was being used for both $|\mathbf{V}_{xz}|$ and $|\mathbf{V}_z|$ should they happen to be the same. Therefore it is useful to know which of the derivatives above is analytically different from the $X_{ref} \neq 0$ case. Obviously, $\partial|\Gamma|^2/\partial|\mathbf{V}_{xz}|$ has vanished, but it might not be so clear which of the others will give the correct value if computed as for $X_{ref} \neq 0$ and which need to be handled by a program branch. Note therefore that when $\mathbf{V}_{xz} \equiv \mathbf{V}_z$;

$$\alpha \rightarrow 2|\mathbf{V}_z|^2 + 2|\mathbf{V}_R|^2 - |\mathbf{V}_s|^2$$

and

$$\beta \rightarrow |\mathbf{V}_s|^2$$

It should then be clear that $\partial|\Gamma|^2/\partial|\mathbf{V}_z|$ requires separate evaluation (compare **12.4-d2** with **12.5-d2**), but the remaining two derivatives (**d3** and **d4**) do not.

OO Basic macros for the power reflection coefficient calculations are given below:

```

Function Scalarvolt_PRC(ByVal Vxz as double, Vz as double, VR as double, Vs as double) as double
'Calculates the magnitude of the power reflection coeff.
'Take the square root of the returned value to get the Voltage refl. coeff. magn.
Dim beta as double
beta = Vs^2 + Vz^2 -Vxz^2
Scalarvolt_PRC = 1
if beta = 0 then exit function
Scalarvolt_PRC = ( Vxz^2 +Vz^2 +2*VR^2 -Vs^2 )/beta
end function

Function SV_sigmaPRC(sigmaV as double, offsetV as double, Vxz as double, Vz as double, VR as double, Vs as double, noX as integer) as double
'Calculates SD of power refl. coeff.
'sigmaV is voltage scale error in percent. OffsetV is voltage offset error in Volts.
'returns an absolute SD.
'if Vxz = Vz and noX = 1, assumes Vxz and Vz are the same measurement.
Dim alpha as double, beta as double
beta = Vs^2 + Vz^2 -Vxz^2
SV_sigmaPRC = 1
if beta = 0 then exit function
alpha = Vxz^2 +Vz^2 +2*VR^2 -Vs^2
Dim sigmaVxz as double, sigmaVz as double, sigmaVR as double, sigmaVs as double
sigmaVxz = Vxz*sigmaV/100 + offsetV
sigmaVz = Vz*sigmaV/100 + offsetV
sigmaVR = VR*sigmaV/100 + offsetV
sigmaVs = Vs*sigmaV/100 + offsetV
Dim derVxz as double, derVz as double, derVR as double, derVs as double
if Vxz = Vz and noX = 1 then
  derVxz = 0
  derVz = 4*VR/Vs^2
else
  derVxz = 4*Vxz*(Vz^2 +VR^2)/beta^2
  derVz = 4*Vz*(Vs^2 -Vxz^2 -VR^2)/beta^2
endif
derVR = 4*VR/beta
derVs = -4*Vs*(Vz^2 + VR^2)/beta^2
SV_sigmaPRC = sqr( (sigmaVxz*derVxz)^2 +(sigmaVz*derVz)^2 +(sigmaVR*derVR)^2 _
+(sigmaVs*derVs)^2 )
end function

```

12a. Error function for the voltage reflection coefficient

The scalar voltage method of equation (12.4) gives us the power reflection coefficient magnitude, and the corresponding error function gives us its standard deviation. The PRC magnitude can be converted into the voltage reflection coefficient (VRC) magnitude by the simple expedient of taking the square root, and other calculations such as VSWR and return loss¹⁷ can follow. Unfortunately however, the process of converting the PRC uncertainty into a VRC uncertainty is not so straightforward.

When two quantities are related by a simple transformation, the uncertainty in one can usually be related to the uncertainty in the other using the rate of change, i.e.;

$$\text{if } y = f(x) \quad \text{then} \quad \sigma_y = \sigma_x \, dy/dx$$

What this statement says is: 'the uncertainty in y is the rate of change of y with respect to x multiplied by the uncertainty in x'. This is the standard rule for propagation of errors, but it will fail if the function is not differentiable, or if the uncertainty is large in comparison to the derivative. We can see the problem in this case when we differentiate the VRC with respect to the PRC:

$$|\Gamma| = \sqrt{|\Gamma^2|} = |\Gamma^2|^{1/2}$$

Thus:

$$\frac{d|\Gamma|}{d|\Gamma^2|} = \frac{1}{2} |\Gamma^2|^{-1/2} = \frac{1}{2\sqrt{|\Gamma^2|}} = \frac{1}{2|\Gamma|}$$

This gives:

$$\sigma_{\text{VRC}} = \frac{\sigma_{\text{PRC}}}{2|\Gamma|}$$

According to this expression, σ_{VRC} becomes very large when $|\Gamma|$ is close to zero, and goes to infinity when $|\Gamma| = 0$. This apparent indeterminacy is however false. It arises because the transfer function is not differentiable over the complete range of interest. This is the same as saying that there is no continuous analytical form for the error function. The expression above is not completely useless because it will work when $|\Gamma| \gg 0$, but for $|\Gamma| \approx 0$ it explodes.

A question that might now spring to mind is: 'what happens if we write a formula for $|\Gamma|$ explicitly and construct an error function by differentiating it with respect to each of the input voltages?' The required expression, obtained by taking the square root of (12.4), is shown below:

$$|\Gamma| = \sqrt{\frac{|\mathbf{V}_{xz}|^2 + |\mathbf{V}_z|^2 + 2|\mathbf{V}_R|^2 - |\mathbf{V}_S|^2}{|\mathbf{V}_S|^2 + |\mathbf{V}_z|^2 - |\mathbf{V}_{xz}|^2}} \quad (12.4a)$$

It is also the case, incidentally, that both the numerator and denominator of this equation are always positive for valid input, and so they can be separated without becoming complex:

$$|\Gamma| = \frac{\sqrt{|\mathbf{V}_{xz}|^2 + |\mathbf{V}_z|^2 + 2|\mathbf{V}_R|^2 - |\mathbf{V}_S|^2}}{\sqrt{|\mathbf{V}_S|^2 + |\mathbf{V}_z|^2 - |\mathbf{V}_{xz}|^2}} \quad (12.4b)$$

¹⁷ Return loss is the reciprocal of the VRC magnitude expressed in dB.

This can be differentiated fairly easily using the quotient rule and the chain (function of a function) rule, but there is still a problem. The numerator of (12.4b) goes to zero when $|\Gamma| \rightarrow 0$. Also due to the application of the chain rule and the fact that $d(x^{1/2})/dx = 1/2x^{-1/2}$, the numerator ends up in the denominator of every derivative, and all of the derivatives explode when $|\Gamma| \rightarrow 0$. What has happened is that we have merely derived the same error function as before, but by a much more long-winded method. The problem needs to be attacked in a completely different way.

A generally effective approach to statistical problems that have no analytical solution is the so-called 'Monte Carlo method'. What the term implies depends on the type of problem, but in this case we can understand it to mean; 'adding noise to the inputs and plotting the distribution of output results after numerous repeated calculations'. Examining the expression for $|\Gamma|$ given above as (12.4a), it should be obvious that if we were to add Gaussian (i.e., normally distributed, random) noise to each of the input voltages, with a half-width equal to the standard deviation in each case; then, after a large number of trial calculations, the output results would be scattered in the approximate form of a normal distribution, with a half-width equal to the desired σ_{VRC} .

Monte Carlo methods however are computationally intensive, and are usually applied to problems that are much more complicated than this. For the present application, we can obtain an approximation for σ_{VRC} by using a related but extremely abbreviated method. We can simply move each of the four input voltages independently to the upper and lower half-width points of their error distributions, record the average output perturbation magnitude in each case and then add the perturbations orthogonally. In other words, we can use the incremental method described in section 7e.

A Basic macro program for calculating σ_{VRC} is given below. It uses both the incremental and the derivative methods and accepts the smaller of the two results. The accuracy of the scalar voltage method for $|VRC|$ is discussed at the end of the next section.

```
Function SV_sigmaVRC(sigmaV as double, offsetV as double, Vxz as double, _
Vz as double, VR as double, Vs as double, noX as integer) as double
'Calculates SD of voltage refl. coeff. Calls the function Scalarvolt_VRC().
'sigmaV is voltage scale error in percent. OffsetV is voltage offset error in Volts.
'Returns an absolute SD.
'if Vxz = Vz and noX = 1, assumes Vxz and Vz are the same measurement.
Dim uVxz as double, uVz as double, uVR as double, uVs as double
'calculate the absolute voltage standard deviations
uVxz = Vxz*sigmaV/100 + offsetV
uVz = Vz*sigmaV/100 + offsetV
uVR = VR*sigmaV/100 + offsetV
uVs = Vs*sigmaV/100 + offsetV
'check that no /0 errors will occur. Max. poss. uncertainty of VRC is 1.
SV_sigmaVRC = 1
if Vs = 0 then exit function
if (Vs+uVs)^2 +vz^2 +Vxz^2 = 0 then exit function
if (Vs-uVs)^2 +vz^2 +Vxz^2 = 0 then exit function
if Vs^2 +(Vz+uVz)^2 +Vxz^2 = 0 then exit function
if Vs^2 +(Vz-uVz)^2 +Vxz^2 = 0 then exit function
if Vs^2 +Vz^2 +(Vxz+uVxz)^2 = 0 then exit function
if Vs^2 +Vz^2 +(Vxz-uVxz)^2 = 0 then exit function
'Calculate the error contributions numerically using the function Scalarvolt_VRC().
Dim DVxz as double, DVz as double, DVR as double, DVs as double, sdnVRC as double
if Vxz = Vz and noX = 1 then
```

```

DVxz = 0
DVz = ( Scalarvolt_VRC(Vz+uVz, Vz+uVz, VR, Vs) _
- Scalarvolt_VRC(Vz-uVz, Vz-uVz, VR, Vs) )/2
else
DVxz = (Scalarvolt_VRC(Vxz+uVxz, Vz, VR, Vs)-Scalarvolt_VRC(Vxz-uVxz, Vz, VR, Vs))/2
DVz = (Scalarvolt_VRC(Vxz, Vz+uVz, VR, Vs)-Scalarvolt_VRC(Vxz, Vz-uVz, VR, Vs))/2
end if
DVR = (Scalarvolt_VRC(Vxz, Vz, VR+uVR, Vs)-Scalarvolt_VRC(Vxz, Vz, VR-uVR, Vs))/2
DVs = (Scalarvolt_VRC(Vxz, Vz, VR, Vs+uVs)-Scalarvolt_VRC(Vxz, Vz, VR, Vs-uVs))/2
'Calculate the numerical SD and set the function value to equal it.
sdnVRC = sqr(DVxz^2 + DVz^2 + DVR^2 + DVs^2)
SV_sigmaVRC = sdnVRC
'Calculate VRC (Gamma) and exit if it is zero or complex (i.e., accept the numerical SD)
Dim alfa as double, beta as double, Gamma as double
alfa = Vxz^2 + Vz^2 + 2*VR^2 - Vs^2
beta = Vs^2 + Vz^2 - Vxz^2
if alfa <= 0 then exit function
if beta <= 0 then exit function
Gamma = sqr(alfa/beta)
'Calculate the analytical SD. Routine is the same as for the PRC SD, with conversion at the end.
Dim derVxz as double, derVz as double, derVR as double, derVs as double
if Vxz = Vz and noX = 1 then
derVxz = 0
derVz = 4*VR/Vs^2
else
derVxz = 4*Vxz*(Vz^2 + VR^2)/beta^2
derVz = 4*Vz*(Vs^2 - Vxz^2 - VR^2)/beta^2
endif
derVR = 4*VR/beta
derVs = -4*Vs*(Vz^2 + VR^2)/beta^2
Dim sdPRC as double, sdVRC as double
sdPRC = sqr( (uVxz*derVxz)^2 +(uVz*derVz)^2 +(uVR*derVR)^2 +(uVs*derVs)^2 )
sdVRC = sdPRC /(2*Gamma)
'If the analytical SD is smaller than the numerical SD, accept the analytical value.
if sdVRC < sdnVRC then SV_sigmaVRC = sdVRC
end function

```

Function **Scalarvolt_VRC**(ByVal **Vxz** as double, **Vz** as double, **VR** as double, **Vs** as double) as double

'Calculates the magnitude of the voltage reflection coeff.

Dim alfa as double, beta as double

Scalarvolt_VRC = 1

beta = Vs^2 + Vz^2 - Vxz^2

if beta <= 0 then exit function

alfa = Vxz^2 + Vz^2 + 2*VR^2 - Vs^2

if alfa < 0 then exit function

Scalarvolt_VRC = sqr(alfa/beta)

end function

Which can be rearranged as follows:

$$\mathbf{V}_B = \frac{\mathbf{V}_S}{m} \left[\frac{\mathbf{Z}_{\text{ref}} - (m-1)\mathbf{Z}}{\mathbf{Z}_{\text{ref}} + \mathbf{Z}} \right]$$

Now, by the simple expedient of setting $m = 2$ we get:

$$\mathbf{V}_B = \frac{\mathbf{V}_S}{2} \left[\frac{\mathbf{Z}_{\text{ref}} - \mathbf{Z}}{\mathbf{Z}_{\text{ref}} + \mathbf{Z}} \right]$$

And if we set $\mathbf{X}_{\text{ref}} = 0$ and $\mathbf{R}_{\text{ref}} = \mathbf{R}_0$:

$$\mathbf{V}_B = \frac{\mathbf{V}_S}{2} \left[\frac{\mathbf{R}_0 - \mathbf{Z}}{\mathbf{R}_0 + \mathbf{Z}} \right]$$

The quantity in square brackets, but for a change of sign, is of course the voltage reflection coefficient. Hence:

$$|\Gamma| = 2 \left| \frac{\mathbf{V}_B}{\mathbf{V}_S} \right| = 2 \frac{|\mathbf{V}_B|}{|\mathbf{V}_S|} \quad (12.7)$$

When the reference reactance can be set to zero, the bridge voltage provides a simple way of obtaining the VRC magnitude. Calculating a standard deviation for a measurement obtained in this way however requires a little thought. It would appear from the expression above, that the uncertainty depends only on the uncertainties in $|\mathbf{V}_B|$ and $|\mathbf{V}_S|$, but that is not the case.

Some readers might have wondered why the parameter m was introduced earlier, when it is obvious that its value must be 2. The answer is that, although we want it to be 2, it is actually determined by the resistors \mathbf{R}_1 and \mathbf{R}_2 , which are only nominally identical. Hence, for the purpose of obtaining the error function, we must rewrite equation (12.7) thus:

$$|\Gamma| = m \frac{|\mathbf{V}_B|}{|\mathbf{V}_S|} = \left(1 + \frac{\mathbf{R}_2}{\mathbf{R}_1} \right) \frac{|\mathbf{V}_B|}{|\mathbf{V}_S|} \quad (12.8)$$

The error function is then:

$$\sigma_{\text{VRC}} = \sqrt{\left(\sigma_{\mathbf{R}_1} \frac{\partial |\Gamma|}{\partial \mathbf{R}_1} \right)^2 + \left(\sigma_{\mathbf{R}_2} \frac{\partial |\Gamma|}{\partial \mathbf{R}_2} \right)^2 + \left(\sigma_{\mathbf{V}_B} \frac{\partial |\Gamma|}{\partial |\mathbf{V}_B|} \right)^2 + \left(\sigma_{\mathbf{V}_S} \frac{\partial |\Gamma|}{\partial |\mathbf{V}_S|} \right)^2}$$

where:

$$\frac{\partial |\Gamma|}{\partial \mathbf{R}_1} = -\frac{\mathbf{R}_2}{\mathbf{R}_1^2} \frac{|\mathbf{V}_B|}{|\mathbf{V}_S|} \quad [\text{Siemens}]$$

$$\frac{\partial |\Gamma|}{\partial \mathbf{R}_2} = \frac{1}{\mathbf{R}_1} \frac{|\mathbf{V}_B|}{|\mathbf{V}_S|} \quad [\text{Siemens}]$$

$$\frac{\partial |\Gamma|}{\partial |\mathbf{V}_B|} = \frac{m}{|\mathbf{V}_s|} \quad [\text{volts}^{-1}]$$

$$\frac{\partial |\Gamma|}{\partial |\mathbf{V}_s|} = -m \frac{|\mathbf{V}_B|}{|\mathbf{V}_s|^2} \quad [\text{volts}^{-1}]$$

For the derivatives of $|\Gamma|$ with respect to the voltages, we are at liberty to put $m = (1 + R_2 / R_1) = 2$ if so desired. Sticking to the value computed from the resistances however makes the method more general; i.e., if some other more obscure quantity having $m \neq 2$ is required, the the routine will still calculate its SD.

In constructing the network, of course, it will be sensible to use two resistors of the same type, and either match them or pick them from the same batch. In that case, both resistances will have the same standard deviation and the error function becomes:

$$\sigma_{\text{VRC}} = \sqrt{\left(\sigma_R \frac{\partial |\Gamma|}{\partial R_1}\right)^2 + \left(\sigma_R \frac{\partial |\Gamma|}{\partial R_2}\right)^2 + \left(\sigma_{\text{VB}} \frac{\partial |\Gamma|}{\partial |\mathbf{V}_B|}\right)^2 + \left(\sigma_{\text{Vs}} \frac{\partial |\Gamma|}{\partial |\mathbf{V}_s|}\right)^2}$$

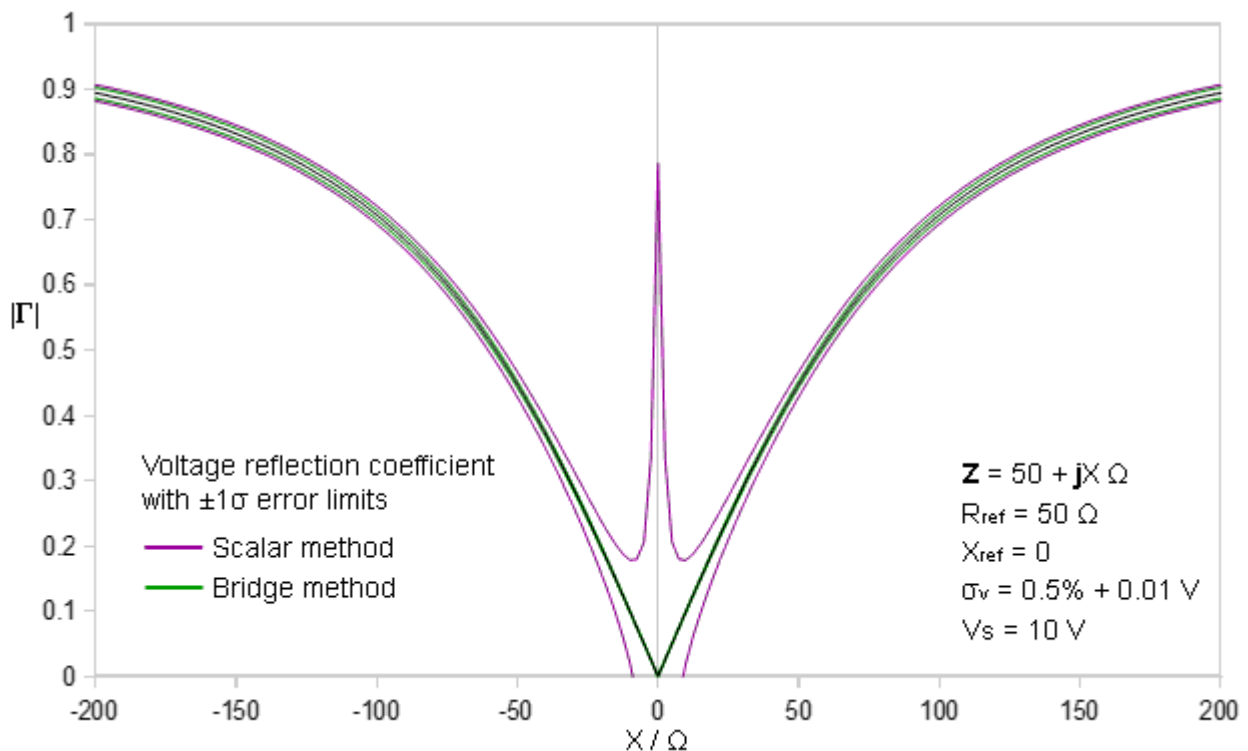
A Basic calculation routine for σ_{VRC} for the bridge method is given below. Note that it calculates an uncertainty for equation (12.8) for any value of m , even though the quantity calculated using (12.8) is only the VRC magnitude when $R_1 = R_2$, $R_{\text{ref}} = R_0$ and $X_{\text{ref}} = 0$.

```
Function BV_sigmaVRC(sigmaR as double, R1 as double, R2 as double, _
sigmaV as double, offsetV as double, VB as double, Vs as double) as double
'calculates SD of voltage reflection coeff. as determined from bridge measurement.
'sigmaR is in percent. R1 and R2 in Ohms.
'sigmaV is voltage scale error in percent. OffsetV is voltage offset error in Volts.
BV_sigmaVRC = 1
if Vs = 0 then exit function
Dim sigmaR1 as double, sigmaR2 as double, sigmaVB as double, sigmaVs as double
sigmaR1 = sigmaR*R1/100
sigmaR2 = sigmaR*R2/100
sigmaVB = VB*sigmaV/100 + offsetV
sigmaVs = Vs*sigmaV/100 + offsetV
Dim m as double, derR1 as double, derR2 as double, derVB as double, derVs as double
m = (R1+R2)/R1
derR1 = -(R2/R1^2)*VB/Vs
derR2 = (1/R1)*VB/Vs
derVB = m/Vs
derVs = -m*VB/Vs^2
BV_sigmaVRC = sqr( (sigmaR1*derR1)^2 +(sigmaR2*derR2)^2 _
+(sigmaVB*derVB)^2 + (sigmaVs*derVs)^2 )
end function
```


12c. Reflectometry error simulation and discussion

A simulation worksheet for the reflection coefficient calculation methods discussed is given in the spreadsheet file [YZ_scalar_sim.ods](#) (sheet 7).

The graph below shows the reflection coefficient for an impedance of $50 + jX \Omega$, as X varies from -200Ω to $+200 \Omega$. The target impedance is $50 + j0 \Omega$. Above and below the principal curve are the $\pm 1\sigma$ error limits for both the scalar voltage method (section 12a) and the bridge method (section 12b). Since comparison of the two methods demands that $X_{\text{ref}} = 0$, the same measurement was used for $|V_{\text{xz}}|$ and $|V_{\text{z}}|$ (i.e., the flag 'noX' was set to 1 for the scalar method error calculation).



The scalar method will always give results that are inferior to the bridge method, because it has to combine more voltage measurements than the bridge method in order to overcome the lack of a facility for vector addition. When $|V_{\text{xz}}|$ and $|V_{\text{z}}|$ are the same however, only three voltages are used, and the accuracy is only a little worse than the bridge method when $|\Gamma|$ is large. Despite the measures taken to prevent the error function from exploding however (see section 12a), the reflection coefficient uncertainty is still poorly determined when the coefficient is small. This does not mean that the value of $|\Gamma|$ obtained will be seriously inaccurate however, it just means that we cannot tell how accurate it is using simple mathematical or computational techniques.

The advantage of the scalar voltage method is that it will work when X_{ref} is not zero. This means that it is possible to make a simultaneous determination of $|\Gamma|$ and all of the other impedance and admittance attributes discussed in this article. Note however, that a necessary condition is that $R_{\text{ref}} = R_0$, whereas most of the measurements that can be carried out have flatter error curves when R_{ref} is somewhat larger than the resistive component of the impedance under test (Z). Hence, best accuracy will always be obtained if the reference configuration is adjusted specifically for the $|\Gamma|$ measurement. If that is to be done however, and particularly since it will be necessary to place a resistance across the generator; we might as well extend the circuit to include the potential divider and the bridge voltage detector.

12d. VSWR calculation

As was mentioned earlier, the voltage (or current) standing wave ratio on a transmission line of characteristic resistance R_0 and of length greater than $\lambda/4$ (electrical) is given by:

$$S = \frac{1+|\Gamma|}{1-|\Gamma|}$$

where $|\Gamma|$ is the voltage reflection coefficient (VRC) magnitude. Thus the accuracy of an SWR determination, assuming that the necessary measurement conditions are met, is:

$$\sigma_s = \sigma_\Gamma \frac{dS}{d|\Gamma|}$$

(where σ_Γ is just another way of writing σ_{VRC}). This expression, of course, assumes that S is differentiable in the region of interest. To differentiate S , we can use the quotient rule, which gives:

$$\frac{dS}{d|\Gamma|} = \frac{1-|\Gamma|+1+|\Gamma|}{(1-|\Gamma|)^2}$$

i.e.,

$$\frac{dS}{d|\Gamma|} = \frac{2}{(1-|\Gamma|)^2}$$

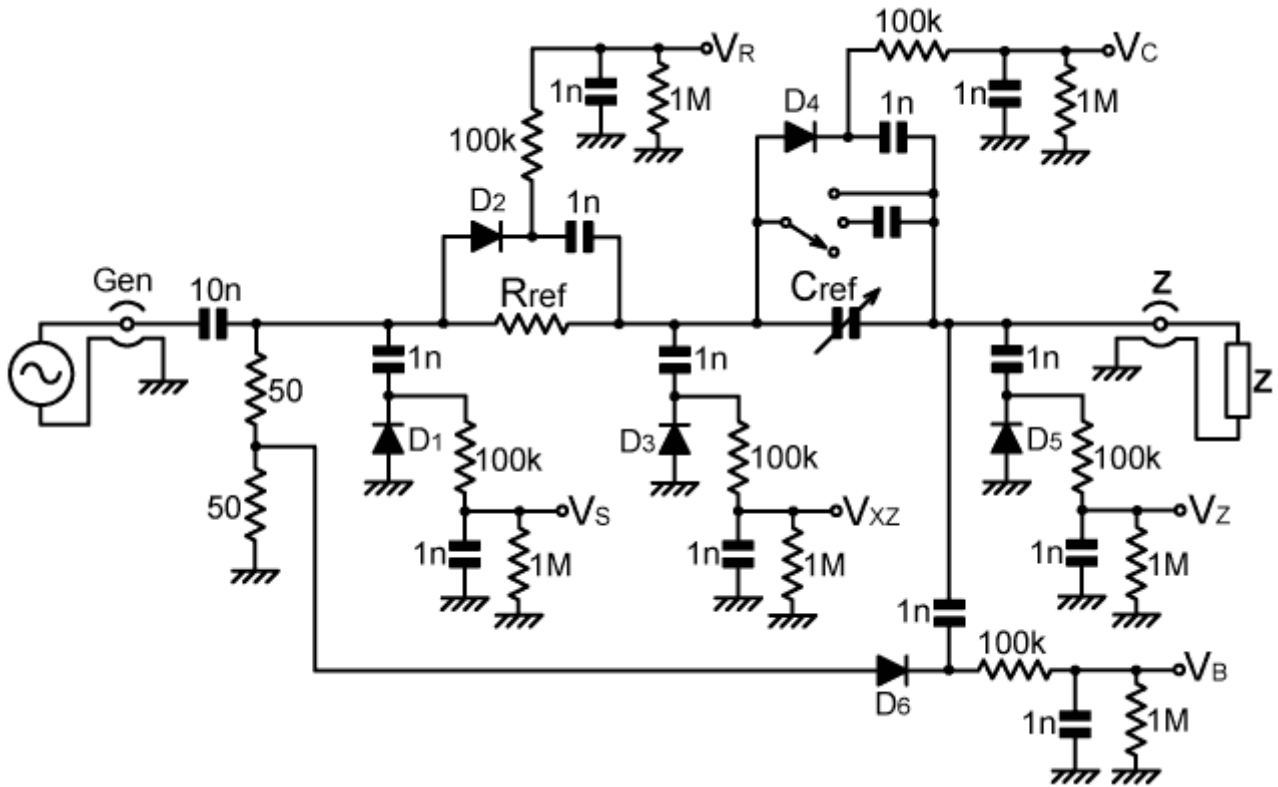
This derivative explodes when $|\Gamma| \rightarrow 1$, i.e., when the transmission line is badly mismatched, but since $S \rightarrow \infty$ when that happens, this is simply a statement to the effect that the absolute uncertainty in S will be large when S is large. The derivative moreover, is perfectly well behaved as $|\Gamma| \rightarrow 0$, which means that we have a viable formula for the standard deviation of a VSWR measurement:

$$\sigma_s = \frac{2\sigma_\Gamma}{(1-|\Gamma|)^2}$$

The only caveat regarding this formula, is that it requires a value of $|\Gamma|$ from a method that produces a well determined standard deviation in the region of $|\Gamma| = 0$. This, of course, is another reason for using the bridge method.

13. Practical issues

Detectors



High impedance shunt diode detectors

Why it is difficult to swap the positions of R_{ref} and X_{ref} . Gen. is terminated in resistor that provides the dc return for V_R and V_X . If we swap them round, we have no guaranteed dc return for V_R .

Must be no DC paths between RF ports and diode circuits - prevents accidental DC injection.
RF isolation issues.

Voltage magnification - need to find the highest voltage in the set for adjusting gen. level (ALC / AGC)

>> Linearising circuits

Diode feedback op-amp method does not correct for dynamic contribution

>> Constant reference reactance

Quasi-constant ref. reactance can be achieved by servoing $|X_{ref}|$ to be some multiple of R_{ref} (such as $1\times$ or $4\times$).

Capacitor model with series partial inductance.

A 2×500 pF broadcast variable capacitor with switched 1000 pF in parallel covers 6 octaves (HF and low VHF) at $X = -50 \Omega$.

>> Analogue computers?

Use of log amps and / or 4-quadrant multipliers to do the calculations.
Complicated & somewhat expensive. Obsolete method?

>> Microcontroller.

A-D conversion,

Built in 10-bit with multi-channel mux in the easy to use types. >0.1% offset error.

Built in USART for taking data from external 12-bit A-D.

>> Match meter R, G and $\tan\phi$

For 50 Ω system:

R measurement ([section 2a](#)) \rightarrow best accuracy when $R_{\text{ref}} \geq 200 \Omega$. $|X_{\text{ref}}| \leq 50 \Omega$ is acceptable.

G measurement ([section 8a](#)) \rightarrow best accuracy when $R_{\text{ref}} \geq 200 \Omega$. $|X_{\text{ref}}| \leq 50 \Omega$ is acceptable.

$\tan\phi$ measurement ([section 7c](#)) \rightarrow best accuracy when $R_{\text{ref}} \geq 200 \Omega$. $X_{\text{ref}} = -50 \Omega$ works well.

Revision history:

v 0.03, 2016-07-13: Equations typeset in OO Math. Some minor corrections.

