Low-frequency effective radius of a single-layer solenoid

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Abstract

Due to path-length variation and strain, the current distribution in the wire of a solenoid inductor is not uniform at low frequencies. This causes the effective solenoid radius to differ from the physical average, leading to a systematic error in inductance calculations. Formulae and and numerical algorithms are given for the effective radius of coils wound with solid rectangular and round wire.

1. Introduction

As has been pointed out Fraga et al. and others[1 2]; the current distribution in the wire of a solenoid inductor is not uniform at low frequencies. The principal reason is that the length of the conduction path is greater on the outside of the coil than it is on the inside. A somewhat less obvious reason is that the act of winding the wire into a helix causes the metal on the outside to stretch, thereby giving rise to a resistivity gradient. These effects are minor when the diameter of the coil is large relative to the thickness of the wire, but can become significant as the curvature of the wire increases.

The overall effect of strain and curvature is to concentrate the conduction current in that part of the wire closest to the inside of the coil. This is a purely resistive phenomenon, as distinct from the complex skin and proximity effects which occur at high frequencies; but the consequence, as in the high-frequency case, is that the diameter of the equivalent current sheet is smaller than the average diameter of the coil. This causes a systematic error when inductance calculations are based on the diameter taken from wire-centre to wire-centre (exacerbated because inductance is proportional to the solenoid radius squared), and may explain why such calculations tend to come out a little too high.

In this article, methods for determining the equivalent current-sheet solenoid radius (r_0) for coils wound with both rectangular and round wire are given. In both instances, consideration is given to the case when the wire is homogeneous (i.e., of uniform resistivity throughout its cross-section), and also to the case when the wire is strained (i.e., has resistivity which increases with distance from the coil axis).

Note that, from consideration of field boundary conditions, r_0 is defined as: 'that distance, measured perpendicular to the coil axis, at which the conduction current flowing on the outside is equal to the current flowing within'. This is a reasonable assumption in relation to practical wound inductors; but it will not necessarily be be valid in extreme cases where the conductor thickness approaches or exceeds the average helix diameter, i.e., when the structure is better described as a machine screw.

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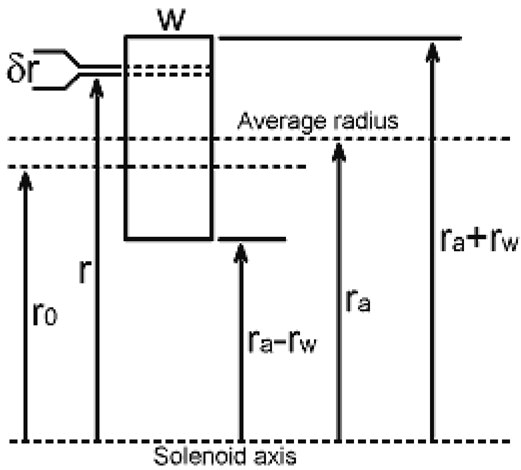
1 **"Practical Model and Calculation of AC resistance of Long Solenoids"**. E. Fraga, C Prados, and D.-X Chen. IEEE Transactions on Magnetics, Vol 34, No. 1. Jan 1998.

2 **Formula for the inductance of a helix made with wire of any section**. C. Snow. BS Sci. 537, 1926. p460. [Available from http://g3ynh.info/zdocs/magnetics/]

2. Homogeneous rectangular conductor

The diagram on the right represents the cross-section of a rectangular wire forming part of a solenoid of average radius r_a. The radial thickness of the wire is defined as $2r_w$ (to be consistent with later analysis of round wire coils), and its axial thickness is w. The radius co-ordinate is r, and the conduction current in the solenoid is distributed over the range:



$r_a - r_w \le r \le r_a + r_w$.

Now consider a slice of conductor, of infinitesimal thickness δr, parallel to the coil axis. The resistance per turn of this slice is:

$R_{t(slice)} = \rho\ \ell_{t(r)} / (\text{ w } \delta r)$

where ρ is the bulk resistivity, and $\ell_{t(r)}$ is the length per turn at radius r. For coils of small pitch angle, the turn length is approximately equal to the solenoid circumference, i.e.:

$\ell_{t(r)} = 2\pi\ r$

Hence:

$R_{t(slice)} = \rho\ 2\pi\ r / (\text{ w } \delta r)$

The current flowing in the slice is inversely proportional to its resistance, i.e., after dropping all constants from the proportionality relationship:

$I_{(slice)}\ \alpha\ (1/r)\ \delta r$

The total conduction current in the solenoid is the sum of the currents flowing in the set of all infinitesimal slices of equal thickness which can exist in the range $r_a-r_w \le r \le r_a+r_w$. Thus:

$$I\ \alpha \int_{r_a-r_w}^{r_a+r_w} \frac{dr}{r}$$

Now, to find the equivalent current sheet radius r_0 , we define it as the distance from the coil axis at which the current flowing outside is equal to the current flowing inside. Hence:

$$\int_{r_a-r_w}^{r_0} \frac{dr}{r} = \int_{r_0}^{r_a+r_w} \frac{dr}{r} \qquad \text{where} \quad \int \frac{dr}{r} = \ln(r) + c$$

i.e.:

$\ln(r_0) - \ln(r_a-r_w) = \ln(r_a+r_w) - \ln(r_0)$

Combining the logarithms:

$\ln[\, r_0/(r_a - r_w) \,] = \ln[\, (r_a + r_w)/r_0 \,]$

and taking the antilog. of both sides:

$r_0/(r_a - r_w) = (r_a + r_w)/r_0$

i.e.:

$r_0^2 = (r_a + r_w)(r_a - r_w) = r_a^2 - r_w^2$

Rearranging this to express the difference between r_0 and r_a as a correction factor gives:

$r_0 = r_a \sqrt{[1 - (r_w/r_a)^2]}$	Equivalent current-sheet radius. Homogeneous rectangular wire. $2\pi r_a >> p$	**2.1**

Note that in deriving the expression above, the length of the conductor at radius r was approximated as the circumference of the solenoid at that radius. Such an approximation is justifiable in many inductor modelling situations, but it is interesting to consider its effect when $r_w = r_a$. It is impossible to wind wire on a coil-former in such a way that the average solenoid radius is equal to half the radial wire thickness, because to do so would require a former of zero radius. Such structures can however be obtained by twisting two wires together and then removing one of the wires, and they can be machined out of solid rod; i.e., the situation $r_a = r_w$ represents the transition from coils to screws. That being the case, we may observe that equation (**2.1**) has an incorrect boundary condition because it predicts $r_0 = 0$ when $r_a = r_w$, whereas (say) a machine screw, due to its surface convolution, must have greater inductance than a straight rod.

The false boundary condition arises because the model allows the conduction path, and hence the resistance, to go to zero at the coil axis when $r_a = r_w$. This does not happen in reality, because a solenoid has finite length. For every turn around the former, the wire advances along the axis by a distance p, where p is known as the 'pitch distance'. Hence, using Pythagoras's theorem, an exact expression for the length per turn of the conduction path at radius r is:

$\ell_{t(r)} = \sqrt{[(2\pi r)^2 + p^2]}$

The resistance per turn of an infinitesimal slice parallel to the axis is now:

$R_{t(slice)} = \rho \, \{\sqrt{[(2\pi r)^2 + p^2]}\,\} \,/\, (\, w \, \delta r \,)$

To simplify the working from now on, let us define a new pitch parameter:

$q = p \,/\, 2\pi$

Thus:

$R_{t(slice)} = \rho \, 2\pi[\sqrt{(r^2 + q^2)}] \,/\, (\, w \, \delta r \,)$

The current in the slice is inversely proportional to the resistance; and so, dropping all constant factors:

$I_{(slice)}\ \alpha\ \delta r / \sqrt{(r^2 + q^2)}$

The equivalent current sheet radius is now defined by the relationship:

$$\int_{r_a-r_w}^{r_0} \frac{dr}{\sqrt{(r^2+q^2)}} = \int_{r_0}^{r_a+r_w} \frac{dr}{\sqrt{(r^2+q^2)}}$$

where[3] [4] $$\int \frac{dr}{\sqrt{(r^2+q^2)}} = \ln[r + \sqrt{(r^2+q^2)}] + c$$

Hence:

$$\ln[r_0+\sqrt{(r_0^2+q^2)}] - \ln[r_a-r_w+\sqrt{\{(r_a-r_w)^2+q^2\}}] = \ln[r_a+r_w+\sqrt{\{(r_a+r_w)^2+q^2\}}] - \ln[r_0+\sqrt{(r_0^2+q^2)}]$$

Rearranging and combining logarithms (bearing in mind that doubling a logarithm is equivalent to squaring its argument), then taking the antilog. of both sides gives:

$$[r_0+\sqrt{(r_0^2+q^2)}]^2 = [\ r_a+r_w+\sqrt{\{(r_a+r_w)^2+q^2\}}\ ]\ [\ r_a-r_w+\sqrt{\{(r_a-r_w)^2+q^2\}}\ ]$$

i.e.:

$$r_0+\sqrt{(r_0^2+q^2)} = \sqrt{[\ (r_a+r_w+\sqrt{\{(r_a+r_w)^2+q^2\}}\ )\ (r_a-r_w+\sqrt{\{(r_a-r_w)^2+q^2\}}\ )\ ]}$$

The right hand side of this expression is composed entirely of constants (i.e., model parameters). Thus, if we define:

$$\alpha^2 = [\ r_a+r_w+\sqrt{\{(r_a+r_w)^2+q^2\}}\ ]\ [\ r_a-r_w+\sqrt{\{(r_a-r_w)^2+q^2\}}\ ]$$

then

$$r_0+\sqrt{(r_0^2+q^2)} = \alpha$$

Subtracting r_0 from each side and then squaring gives:

$$r_0^2+q^2 = (\alpha - r_0)^2$$

Expanding the right hand side gives:

$$r_0^2+q^2 = \alpha^2 -2\alpha\ r0 + r_0^2$$

i.e.:

$$r_0 = (\alpha^2 - q^2) / (\ 2\alpha\ )$$

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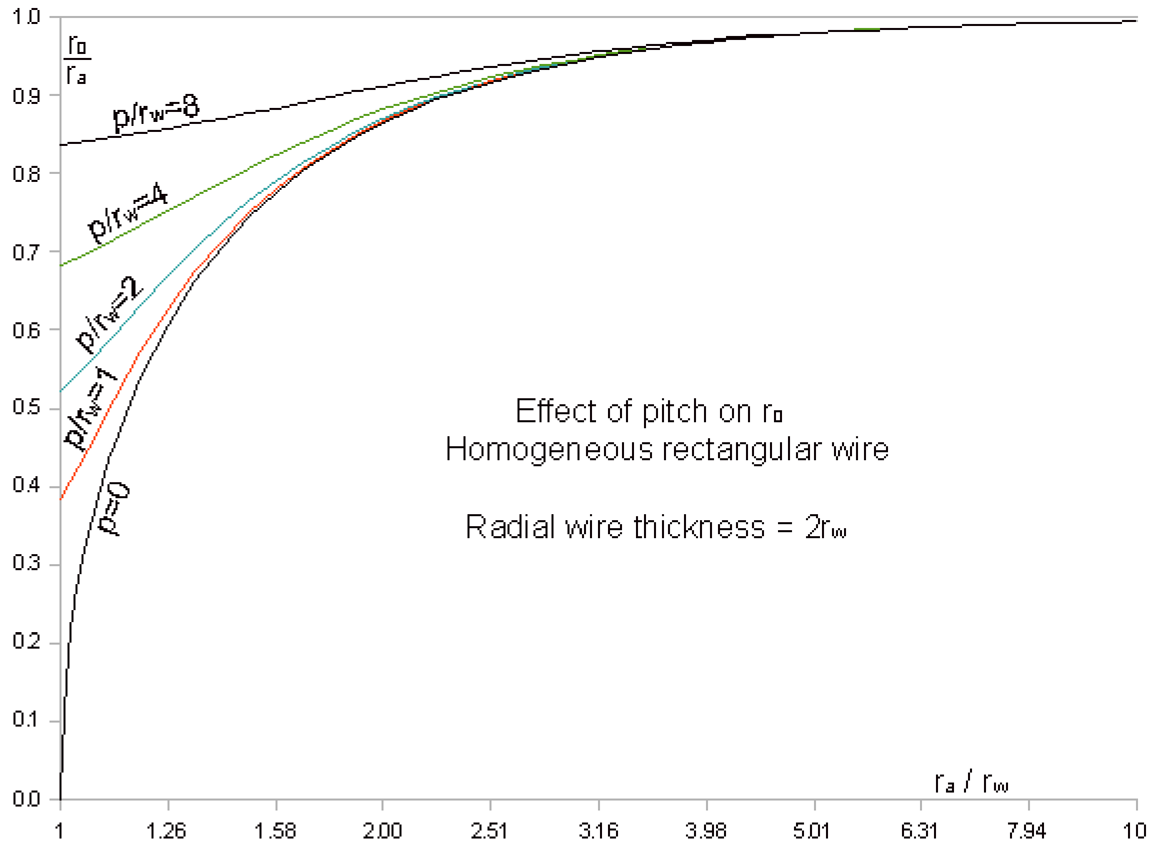
3 **<u>Tables of Series, Products, and Integrals</u>**. I M **Ryshik** and I S **Gradstein**. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957. Sect. 2.27; 2.271-4, also: sect. 2.26; 2.261.

4 **<u>Tables of Integrals and Other Mathematical Data</u>**. H B **Dwight**. 4th edn. Macmillan 1961. Library of Congrress cat. 61-6419. Formula 200.01 (p50).

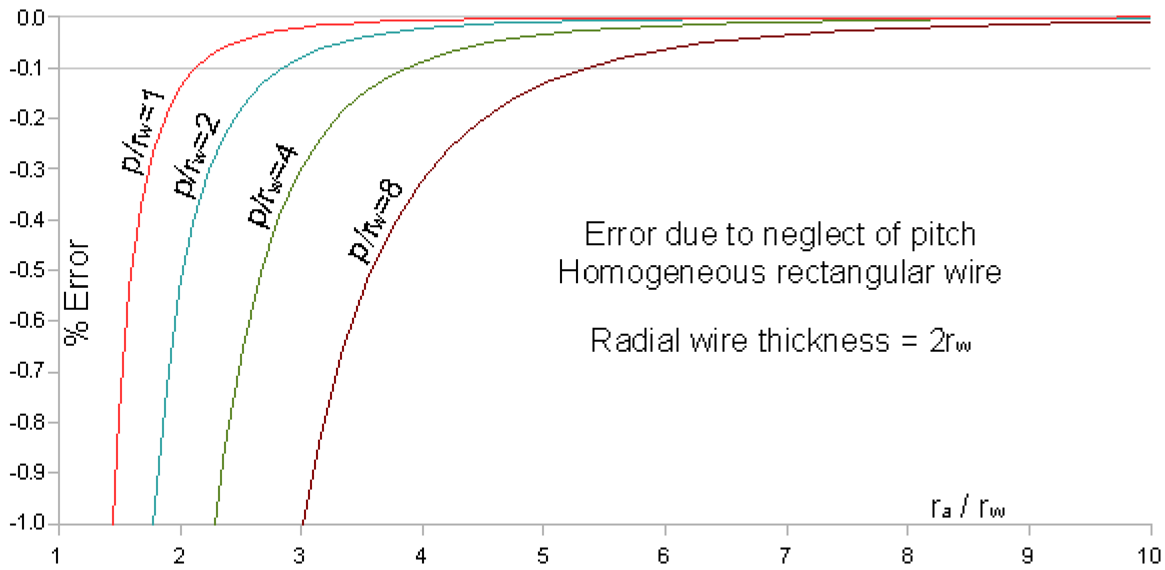
Hence:

$r_0 = (\alpha/2) - (q^2/2\alpha)$	Equivalent current-sheet radius. Homogeneous rectangular wire	**2.2**
where: $\alpha^2 = [r_a+r_w+\sqrt{\{(r_a+r_w)^2+q^2\}}] [r_a-r_w+\sqrt{\{(r_a-r_w)^2+q^2\}}]$ and: $q = p/2\pi$		

The effect of pitch on r_0 is shown in the graph below, where coil parameters are expressed in units of r_w [Spreadsheet calculation: **r0_rect.ods** (sheet 1)]. Note that the data have been plotted on a logarithmic r_a/r_w scale to expand the region where pitch is significant.



For pitch values up to $8r_w$, equation (**2.2**) is convergent with (**2.1**) to within 0.07% for $r_a > 6r_w$. In practical terms, $r_a = 6r_w$ might correspond to (say) a wire of 2mm radial thickness wound onto a 10mm diameter former, i.e., a tightly wound coil. The actual error in r_0 incurred by using the approximation (**2.1**) instead of the exact expression (**2.2**) is plotted below.



Note that, unless machined out of solid bar, the coils for which equation (**2.1**) is inadequate cannot be constructed without deforming the wire considerably. It follows that, for conventional solenoids, neither model is strictly valid; and accuracy in calculating the inductance of tightly wound coils will be improved by taking the effect of wire strain into account.

3. Strained rectangular conductor

When wire is wound into a coil, the strain induced by so doing will give rise to a resistivity gradient in the conductor cross-section. That this is realistic is evident from the resistivity data for copper in the Rubber Handbook[5], which gives a figure of 17.241 nΩm for annealed Cu and 17.71 nΩm for hard-drawn Cu (both measurements at 20°C). In order to model the strain effect, we will assume that the increase in resistivity is proportional to the relative elongation.

Referring again to the diagram at the beginning of section **2**; observe that when $2\pi r_a >> p$, the resistance per turn of an infinitesimal slice can be written:

$$R_{t(slice)} = \rho_r \, 2\pi r \, / \, (w \, \delta r)$$

where ρ_r is the resistivity at raduis r. If we take the resistivity at the average radius r_a to be ρ_a ; then, bearing in mind that the elongation is proportional to the turn length at radius r (i.e., the circumference when pitch is neglected), the resistivity at radius r is:

$$\rho_r = \rho_a \, r \, / \, r_a$$

The choice of the average radius as the resistivity reference point is, incidentally, arbitrary (and inconsequential in the present context). In reality, if we assume that the metal is effectively incompressible, then the material closest to the inside of the coil will have resistivity unchanged from its original value, and everything outside that radius will have been stretched. Whether we use r_a or r_a-r_w as the reference radius however makes no difference, because either choice produces a

5 **CRC Handbook of Chemistry and Physics**, 63rd edition. 1982-83 (CRC press, Florida) [Newer editions exist]. E81: Resistivity of metals.

constant parameter which will cancel from the integral equation which defines r_0. Hence, sticking with r_a as the reference:

$R_{t(slice)} = \rho_a \, 2\pi r^2 / (r_a \, w \, \delta r)$

As before, the current in the slice is inversely proportional to its resistance; and so, dropping constants:

$I_{(slice)} \;\alpha\; (1/r^2)\delta r$

Hence, for p = 0 or $2\pi r_a >> p$, the equivalent current sheet radius can be obtained from the relationship:

$$\int_{r_a-r_w}^{r_0} \frac{dr}{r^2} = \int_{r_0}^{r_a+r_w} \frac{dr}{r^2}$$

In this case, integration is simply the reverse of differentiation, and so:

$$\frac{-1}{r_0} + \frac{1}{r_a-r_w} = \frac{-1}{r_a+r_w} + \frac{1}{r_0}$$

i.e.;

$$\frac{2}{r_0} = \frac{1}{r_a+r_w} + \frac{1}{r_a-r_w} = \frac{r_a - r_w + r_a + r_w}{(r_a+r_w)(r_a-r_w)} = \frac{2\,r_a}{r_a^2 - r_w^2}$$

Taking the reciprocal of both sides gives:

$r_0 = (r_a^2 - r_w^2) / r_a$

and by factoring r_a^2 from the numerator we get:

$r_0 = r_a [1 - (r_w/r_a)^2]$	Equivalent current-sheet radius. Strained rectangular wire. $2\pi r_a >> p$	**3.1**

As expected, the effective current sheet radius is reduced as a consequence of strain, the difference between equation (**3.1**) and equation (**2.1**) being the loss of a square-root symbol operating on the radius correction factor.

Equation (**3.1**), of course, also has a false boundary condition at $r_a = r_w$, and we can correct that, as before, by taking the length per turn at radius r as $\sqrt{[(2\pi r)^2 + p^2]}$, i.e.;

$\ell_{t(r)} = 2\pi\sqrt{[r^2 + q^2]}$

where $q = p/2\pi$.

The resistivity relative to its value at r_a (say) is proportional to the relative elongation, i.e.;

$\rho_r = \rho_a \{\sqrt{[r^2 + q^2]}\} / \{\sqrt{[r_a^2 + q^2]}\}r$

Thus:

$R_{t(slice)} = \rho_r \, \ell_{t(r)} / (w\,dr)$

i.e.:

$R_{t(slice)} = \rho_a \, 2\pi \, (r^2 + q^2) / [\, w\, \delta r \, \sqrt{(r_a^2 + q^2)}\,]$

Hence, noting that $(r_a^2 + q^2)$ is a constant:

$I_{(slice)} \;\alpha\; \delta r / (r^2 + q^2)$

and so, making the total currents inside and outside r_0 the same, we get:

$$\int_{r_a - r_w}^{r_0} \frac{dr}{r^2 + q^2} = \int_{r_0}^{r_a + r_w} \frac{dr}{r^2 + q^2}$$

where[6], for q>0 $\displaystyle\int \frac{dr}{r^2 + q^2} = (1/q)\,\mathrm{Arctan}(r/q) + c$

In this case, there is no solution for $q = 0$ (because $1/q \to \infty$) but we already have that from equation (**3.1**). Hence, for $q > 0$;

$(1/q)\,\mathrm{Arctan}(r_0/q) - (1/q)\,\mathrm{Arctan}\{\,(r_a - r_w)/q)\,\} = (1/q)\,\mathrm{Arctan}\{\,(r_a + r_w)/q)\,\} - (1/q)\,\mathrm{Arctan}(r_0/q)$

i.e.:

$2\,\mathrm{Arctan}(r_0/q) = \mathrm{Arctan}\{\,(r_a + r_w)/q)\,\} + \mathrm{Arctan}\{\,(r_a - r_w)/q)\,\}$

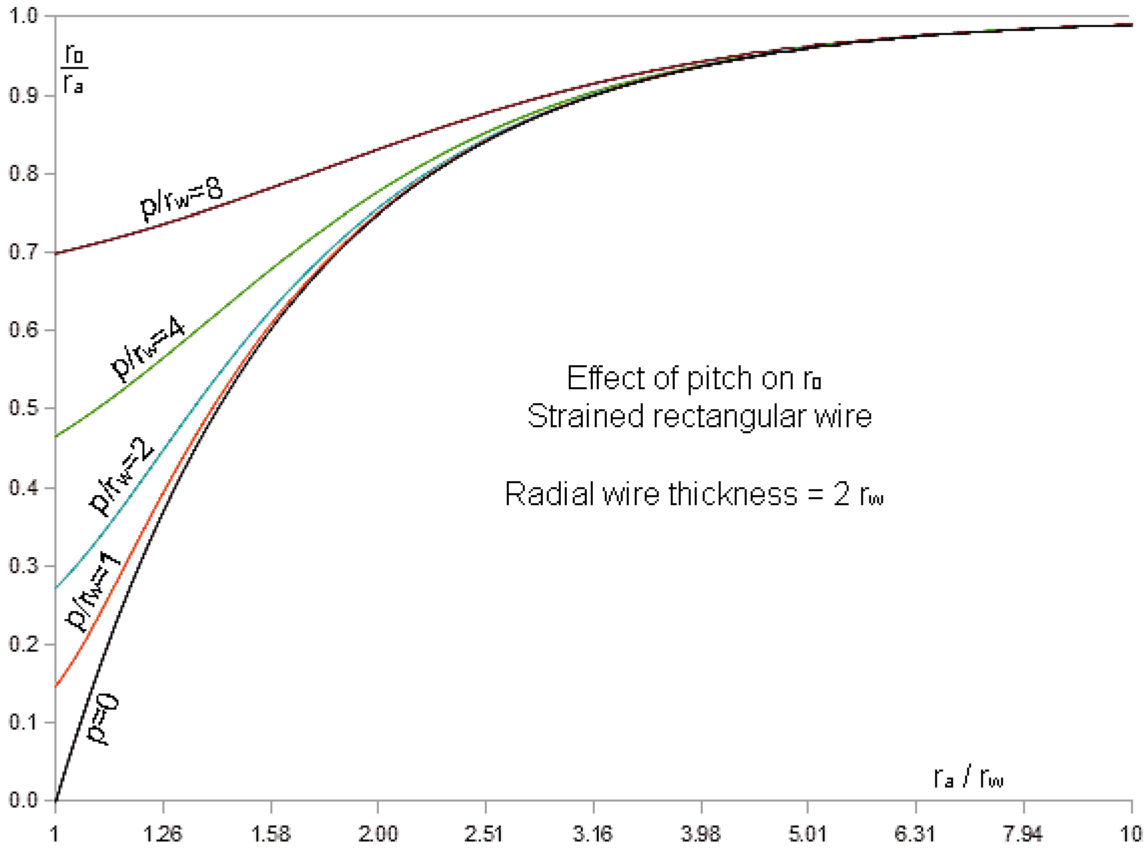
Hence:

$r_0 = q\,\mathrm{Tan}[\tfrac{1}{2}\mathrm{Arctan}\{(r_a + r_w)/q)\} + \tfrac{1}{2}\mathrm{Arctan}\{(r_a - r_w)/q)\}]$	Equivalent current-sheet radius. Strained rectangular wire $q = p/2\pi > 0$, $r_a > r_w$	**3.2**

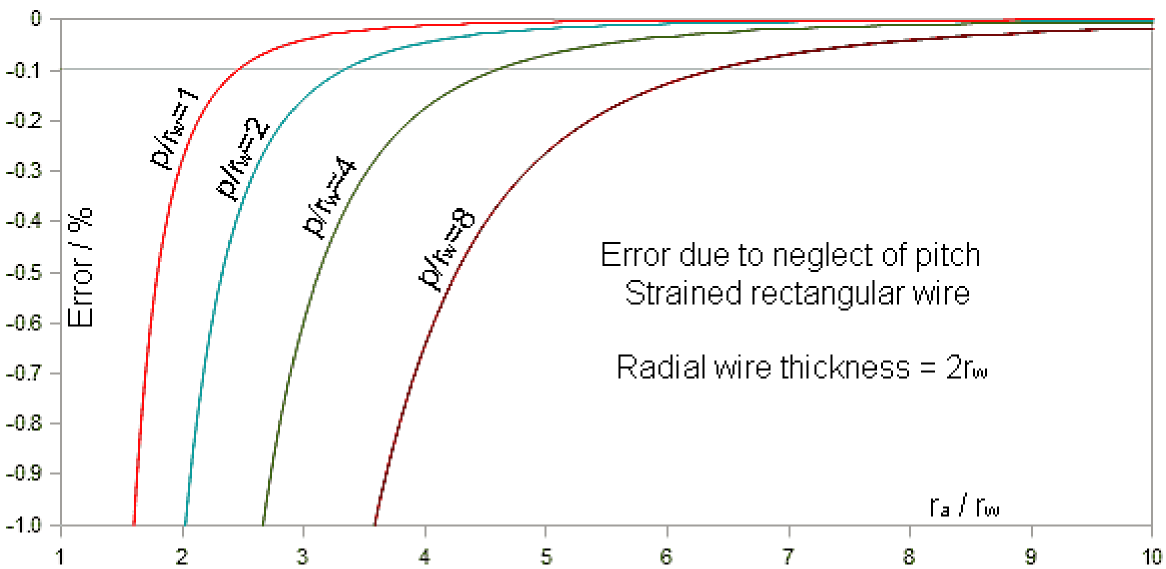
Note that there is a possible substitution for Arctan(a)+Arctan(b) (see for example, Ryshik and Gradstein 1.625-8) but the alternative form encounters infinities within the allowed argument range. Hence (**3.2**) is the best continuous solution for p>0.

6 **Ryshik & Gradstein**. 2.124-1. **Dwight** 120.1

The effect of pitch on r_0 for the strained wire case is shown below, with coil parameters expressed in units of r_w as before [Spreadsheet calculation: **r0_rect.ods** (sheet 2)].



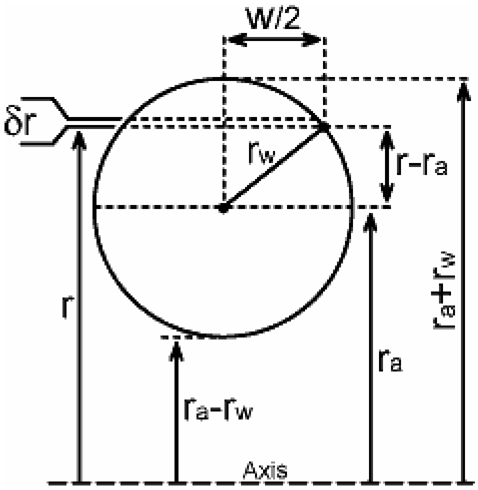
The error in r_0 incurred by using the approximation (**3.1**) instead of the exact expression (**3.2**) is shown in the graph below.



4. Homogeneous round conductor

Deriving expressions for the effective radius of a round wire solenoid is considerably more difficult than it is for the rectangular wire case. The resulting integrals do however have analytical solutions for the cases in which pitch can be neglected (i.e., when p=0); and although those solutions do not lead to closed-form expressions for r_0, they nevertheless allow it to be evaluated iteratively.

Starting with the homogeneous case; consider (as before) an infinitesimal slice of conductor parallel to the solenoid axis. The resistance per turn of this slice, for the case when $2\pi r_a >> p$, is :



$R_{t(slice)} = \rho\ 2\pi r / (\ w\ \delta r\ )$

For a round wire however, the relationship is complicated by the fact that the width of the slice (w) varies as a function of r. Using Pythagoras's theorem:

$(w/2)^2 = r_w^2 - (r-r_a)^2$

i.e.:

$w = 2\ \sqrt{[r_w^2 - (r-r_a)^2]}$

Hence:

$R_{t(slice)} = \rho\pi r / (\sqrt{[r_w^2 - (r-r_a)^2]}\ \delta r\ )$

The current in the slice is inversely proportional to the resistance; and so, dropping all constant factors:

$I_{(slice)}\ \alpha\ (1/r)\ \sqrt{[r_w^2 - (r-r_a)^2]}\ \delta r$

The equivalent current sheet radius r_0 must now be extracted from the relationship:

$$\int_{r_a-r_w}^{r_0} \frac{\sqrt{[r_w^2 - (r-r_a)^2]}\ dr}{r} = \int_{r_0}^{r_a+r_w} \frac{\sqrt{[r_w^2 - (r-r_a)^2]}\ dr}{r} \qquad (4.1)$$

The piecewise solution for this integral is given by Ryshik and Gradstein (section 2.26); but in order to apply it, we need to put the problem into standard form according to the conventions of that source. Hence we start by defining a quadratic polynomial:

$s_{(r)} = a + br + cr^2 = r_w^2 - (r-r_a)^2 = r_w^2 - r_a^2 + 2r_a\ r - r^2$

Thus:

$a = -(r_a^2-r_w^2) < 0$

$b = 2r_a$

$c = -1 < 0$

and:

$-\Delta = b^2 - 4ac = 4r_a^2 - 4(r_a^2 - r_w^2) = 4r_w^2 > 0$

The solution is as follows (Ryshik and Gradstein, section 2.26):

$$\int \frac{\sqrt{s_{(r)}}\, dr}{r} = (\sqrt{s_{(r)}}) + a \int \frac{dr}{r\sqrt{s_{(r)}}} + \frac{b}{2} \int \frac{dr}{\sqrt{s_{(r)}}} \qquad \text{[R\&G 2.267-1]}$$

Where, for $a < 0$ and $\Delta < 0$:

$$\int \frac{dr}{r\sqrt{s_{(r)}}} = \frac{-1}{\sqrt{-a}} \text{Arcsin} \left[\frac{2a + br}{r\sqrt{-\Delta}} \right] + k'' \qquad \text{[R\&G 2.266]}$$

and for $c < 0$, $\Delta < 0$ and $s_{(r)} > 0$:

$$\int \frac{dr}{\sqrt{s_{(r)}}} = \frac{-1}{\sqrt{-c}} \text{Arcsin} \left[\frac{2cr + b}{\sqrt{-\Delta}} \right] + k' \qquad \text{[R\&G 2.261]}$$

(where k is a constant of integration). Hence, for $a < 0$, $c = -1$ and $\Delta < 0$:

$$\int \frac{\sqrt{s_{(r)}}\, dr}{r} = (\sqrt{s_{(r)}}) + (\sqrt{-a})\text{Arcsin} \left[\frac{2a + br}{r\sqrt{-\Delta}} \right] - \frac{b}{2} \text{Arcsin} \left[\frac{b - 2r}{\sqrt{-\Delta}} \right] + k$$

Now restoring the original parameters, and noting that a ' ± ' symbol is created each time a square-root is evaluated, we get:

$$\int \frac{\sqrt{s_{(r)}}\, dr}{r} = (\sqrt{s_{(r)}}) \mp [\sqrt{(r_a^2 - r_w^2)}] \text{ Arcsin} \left[\frac{r_a r - (r_a^2 - r_w^2)}{\pm r_w r} \right] \mp r_a \text{ Arcsin} \left[\frac{r_a - r}{\pm r_w} \right] + k$$

This expression can now be substituted into equation (**4.1**), but it will save a considerable amount of confusion of we draw up a table showing the values of the three terms at the upper and lower limits $r = r_a + r_w$ and $r = r_a - r_w$. Simplification of the middle term requires the substitution:

$r_a^2 - r_w^2 = (r_a + r_w)(r_a - r_w)$.

Term	$r = r_a - r_w$	$r = r_a + r_w$
$\sqrt{s_{(r)}} = \sqrt{[r_w^2 - (r-r_a)^2]}$	0	0
$\mp[\sqrt{(r_a^2-r_w^2)}]$ Arcsin$\{ [r_a r-(r_a^2-r_w^2)] / [\pm r_w r] \}$	$\mp[\sqrt{(r_a^2-r_w^2)}] \times -\{\pm\pi/2\}$	$\mp[\sqrt{(r_a^2-r_w^2)}] \times \{\pm\pi/2\}$
$\mp r_a$ Arcsin$[(r_a-r)/\pm r_w]$	$\mp r_a \times \{\pm\pi/2\}$	$\mp r_a \times -\{\pm\pi/2\}$

It is not possible to eliminate the ± operators at this stage, but they are conserved when substitutions are made for r and so can be treated as constants. Hence, referring to the table; it can be seen that the integral evaluated at the lower limit on the left hand side of (**4.1**) is equal to the integral evaluated at the upper limit on the right, and complete cancellation occurs. All that remains is the integral evaluated at r_0 on the left, and the negative of the same quantity on the right. Hence, moving everything to the left side and dividing by 2:

$$[\sqrt{\{r_w^2 - (r_0-r_a)^2\}}] \mp [\sqrt{(r_a^2-r_w^2)}] \text{ Arcsin} \left[\frac{r_a r_0-(r_a^2-r_w^2)}{\pm r_w r_0} \right] \mp r_a \text{ Arcsin} \left[\frac{r_a - r_0}{\pm r_w} \right] = 0$$

The sign ambiguities give four possible ways in which the terms can be combined to produce a sum of zero. For any choice of signs moreover, there is no closed-form analytical solution for r_0 , because instances of it occur both within and outside the arguments of the arcsines. Thus we are faced with the need to guess the correct combination and then devise an iterative calculation procedure.

The obvious first line of attack is to choose values of r_a and r_w and then search for roots on the interval $0 < r_0 < r_a$ for each combination. Unfortunately however, this approach produces at least two physically plausible solutions, and the outcome remains ambiguous. This difficulty was eventually resolved by direct numerical integration of equation (**4.1**), using an algorithm described in section **6**; i.e., the correct choice of signs was determined by putting-in r_0 values obtained by an independent method. The development of a complete alternative approach, of course, casts doubt on the utility of this analysis; but the method given here is faster and inherently more accurate than numerical integration.

The correct combination was found to be:

$$[\sqrt{\{r_w^2 - (r_0-r_a)^2\}}] - [\sqrt{(r_a^2-r_w^2)}] \text{ Arcsin} \left[\frac{r_a r_0-(r_a^2-r_w^2)}{r_w r_0} \right] - r_a \text{ Arcsin} \left[\frac{r_a - r_0}{r_w} \right] = 0 \qquad (\mathbf{4.2})$$

An iteration procedure was developed as follows. Firstly, the objective is to calculate a radius correction factor r_0/r_a , which can be multiplied by r_a in order to obtain r_0. Hence it is appropriate to express all parameters in units of r_a. Thus, if we define the following normalised parameters:

$r_{0/} = r_0/r_a$ and $r_{w/} = r_w/r_a$,

and divide every parameter by r_a we get:

$$[\sqrt{\{r_{w/}^2 - (r_{0/}-1)^2\}}] - [\sqrt{(1-r_{w/}^2)}]\ \text{Arcsin}\left[\frac{r_{0/}-(1-r_{w/}^2)}{r_{w/}\ r_{0/}}\right] - \text{Arcsin}\left[\frac{1-r_{0/}}{r_{w/}}\right] = 0 \qquad (4.3)$$

Finding an iteration formula is now a matter of designating one of the terms as primary, and the other two as secondary. The secondary terms are evaluated using an initially guessed value, $r_{0/}'$ say, and an estimate for $r_{0/}$ is then found using a rearrangement of (4.3). Feeding the latest estimate (or a weighted average of the latest and the prior estimate) back into the formula then produces a refined estimate, and the procedure is repeated until the difference between the current and the previous result becomes very small. Such is the principle; but of course it is necessary to obtain an iteration formula which is convergent and stable; i.e., the estimate must always improve after a round of iteration, and it must do so for any allowed value of the input argument $r_{w/}$.

In this case, after some experimentation using a spreadsheet (**r0_round.ods**, sheet 1), a convergent iteration procedure was found using the first term of (4.3) as primary. Thus if we define the secondary terms as:

$$u = [\sqrt{(1-r_{w/}^2)}]\ \text{Arcsin}\{\ [\ r_{0/}'-(1-r_{w/}^2)\ ]\ /\ (\ r_{w/}\ r_{0/}'\ )\ \}$$

and

$$v = \text{Arcsin}\{\ (\ 1-r_{0/}'\ )\ /\ r_{w/}\ \}$$

where $r_{0/}'$ is the starting estimate, we get:

$$\sqrt{[r_{w/}^2 - (r_{0/}-1)^2]} = u + v$$

i.e.,

$$r_{w/}^2 - (r_{0/}-1)^2 = (\ u + v\ )^2$$

$$(r_{0/}-1)^2 = r_{w/}^2 - (\ u + v\ )^2$$

$$r_{0/}-1 = \pm\sqrt{[\ r_{w/}^2 - (\ u + v\ )^2\ ]}$$

and since $r_{0/}$ lies between 0 and 1; the negative of the square root is the correct choice and the refined estimate for $r_{0/}$ is:

$$r_{0/} = 1 - \sqrt{[\ r_{w/}^2 - (\ u + v\ )^2\ ]}$$

This iteration method is used in the Open Office Basic[7] macro routine shown below, with some small refinements to ensure stability. Graphs produced by the calculation are given in section **7**, and are compared with the numerical integration method in the spreadsheet **r0_round_p.ods** (sheet 1).

Referring to the program code: the initial approximation is:

$$r_{0/}' = 1$$

7 http://wiki.services.openoffice.org/wiki/Documentation/BASIC_Guide/Language

The use of a crude starting value makes little difference to the overall calculation time because the initial rate of convergence is fast. Feeding the new estimate directly into the iteration formula however results in overshoot, and that can lead to program error. The solution is to feed back the average of the current and the previous values of $r_{0/}$, so that:

$$r_{0/(new)} = (r_{0/} + r_{0/}')/2$$

Finally, it was discovered that for very small values of $r_{w/}$, which will in any case give $r_{0/}$=1 to a large number of decimal places, the argument of the square root can become slightly negative (causing a program error) during the first cycle of iteration. The solution is to check to make sure that the argument is > 0 and exit the loop if it is not.

With the convergence test criterion shown, the routine returns r_0/r_a to 12 decimal-place accuracy. The maximum number of iterations required is 37. The function ' Arcsin(x) ' below the main routine is required because the Basic programming language has no library function for the inverse sine.

```
Function Shorad(byval rwa as double) as double
'Returns r0/ra for a solenoid wound with homogeneous cylindrical wire
'Calling argument is rw/ra.  pitch is neglected.
'v1.00. D W Knight, April 2010.
Dim u as double, v as double, x as double, a as double
Dim test as double, r0 as double, r0p as double
if rwa >= 1 then
 Shorad = 0
elseif rwa <= 0 then
 Shorad = 1
else
 r0p = 1
 do
   x = (r0p+rwa*rwa-1)/(r0p*rwa)
   u = sqr(1-rwa*rwa)*Arcsin(x)
   v = Arcsin((1-r0p)/rwa)
   a = rwa*rwa-(u+v)*(u+v)
   if a<0 then
    r0=1
    exit do
   else
    r0 = 1-sqr(a)
   endif
   test=r0-r0p
   r0p=(r0+r0p)/2
 loop until abs(test) < 1E-12
 Shorad=r0p
endif
end function
```

```
Function Arcsin(byval x as double) as double
if x >= 1 then
  Arcsin = pi/2
elseif x <= -1 then
  Arcsin = -pi/2
else
  Arcsin = Atn(x/sqr(-x*x+1))
endif
end function
```

5. Strained round conductor

The principal effect of straining a round wire to make it follow a coil former, is of course, to produce a resistivity gradient as it did for the rectangular wire case. This introduces an r^2 dependence in the denominator of the current integral as before. Hence, the effective solenoid radius for the strained round wire case (neglecting pitch) is captured in the relationship:

$$\int_{r_a-r_w}^{r_0} \frac{\sqrt{[r_w^2 - (r-r_a)^2]}\,dr}{r^2} = \int_{r_0}^{r_a+r_w} \frac{\sqrt{[r_w^2 - (r-r_a)^2]}\,dr}{r^2} \qquad (5.1)$$

Once again, a solution is given by Ryshik and Gradstein, and the polynomial inside the square-root bracket is put into standard form as before:

$s_{(r)} = a + br + cr^2 = r_w^2 - (r-r_a)^2 = r_w^2 - r_a^2 + 2r_a\,r - r^2$

$a = -(r_a^2-r_w^2) < 0$

$b = 2r_a$

$c = -1 < 0$

$-\Delta = b^2 - 4ac = 4r_w^2 > 0$

The solution is (Ryshik and Gradstein, section 2.26):

$$\int \frac{\sqrt{s_{(r)}}\,dr}{r^2} = \frac{-\sqrt{s_{(r)}}}{r} + \frac{b}{2}\int \frac{dr}{r\sqrt{s_{(r)}}} + c\int \frac{dr}{\sqrt{s_{(r)}}} \qquad \text{[R\&G 2.267-2]}$$

Where, for $a < 0$ and $\Delta < 0$:

$$\int \frac{dr}{r\sqrt{s_{(r)}}} = \frac{-1}{\sqrt{-a}}\,\text{Arcsin}\left[\frac{2a + br}{r\sqrt{-\Delta}}\right] + k'' \qquad \text{[R\&G 2.266]}$$

and for $c < 0$, $\Delta < 0$ and $s_{(r)} > 0$:

$$\int \frac{dr}{\sqrt{s_{(r)}}} = \frac{-1}{\sqrt{-c}} \text{ Arcsin} \left[\frac{2cr + b}{\sqrt{-\Delta}} \right] + k' \qquad \text{[R\&G 2.261]}$$

Hence, for $a < 0$, $c = -1$ and $\Delta < 0$:

$$\int \frac{\sqrt{s_{(r)}}\, dr}{r^2} = \frac{-\sqrt{s_{(r)}}}{r} - \frac{b}{2\sqrt{-a}} \text{ Arcsin} \left[\frac{2a + br}{r\sqrt{-\Delta}} \right] \pm \text{Arcsin} \left[\frac{b - 2r}{\sqrt{-\Delta}} \right] + k$$

Restoring the original parameters gives:

$$\int \frac{\sqrt{s_{(r)}}\, dr}{r^2} = \frac{-\sqrt{s_{(r)}}}{r} - \frac{ra}{\sqrt{(r_a^2 - r_w^2)}} \text{ Arcsin} \left[\frac{r_a\, r - (r_a^2 - r_w^2)}{\pm r_w\, r} \right] \pm \text{Arcsin} \left[\frac{r_a - r}{\pm r_w} \right] + k$$

In applying this set of solutions to equation (5.1), we may observe that all of the constant terms cancel, as they did in the homogeneous wire case discussed previously. That leaves the sum of terms containing r_0 to be equal to zero, as before. Thus:

$$\frac{\sqrt{[r_w^2 - (r_0 - r_a)^2]}}{r_0} \pm \frac{r_a}{\sqrt{(r_a^2 - r_w^2)}} \text{ Arcsin} \left[\frac{r_a\, r_0 - (r_a^2 - r_w^2)}{r_w\, r_0} \right] \pm \text{Arcsin} \left[\frac{r_a - r_0}{r_w} \right] = 0$$

As before, the sign ambiguities were resolved by using values of r_0/r_a produced by the numerical integration method described in section **6**. The correct combination was found to be:

$$\frac{\sqrt{[r_w^2 - (r_0 - r_a)^2]}}{r_0} - \frac{r_a}{\sqrt{(r_a^2 - r_w^2)}} \text{ Arcsin} \left[\frac{r_a\, r_0 - (r_a^2 - r_w^2)}{r_w\, r_0} \right] - \text{Arcsin} \left[\frac{r_a - r_0}{r_w} \right] = 0 \qquad (5.2)$$

No closed form analytical solution exists and so r_0 must be calculated iteratively. Dividing all parameters by r_a and using the reduced parameter definitions:

$r_{0/} = r_0/r_a$ and $r_{w/} = r_w/r_a$,

equation (5.2) becomes:

$$\frac{\sqrt{[r_{w/}^2 - (r_{0/} - 1)^2]}}{r_{0/}} - \frac{1}{\sqrt{(1 - r_{w/}^2)}} \text{ Arcsin} \left[\frac{r_{0/} + r_{w/}^2 - 1}{r_{w/}\, r_{0/}} \right] - \text{Arcsin} \left[\frac{1 - r_{0/}}{r_{w/}} \right] = 0 \qquad (5.3)$$

In this case, two stable iteration procedures were found and compared to find the one with the fastest rate of convergence (see spreadsheet: **r0_round.ods**, sheet 2). The best formula is used in the Open Office Basic routine shown below. It uses the first term in (5.3) as primary and is derived as follows.

Let:

$y = [\ 1 / \sqrt{(1 - r_{w/}^2)}\ ] \text{Arcsin}\{\ (\ r_{0/}' + r_{w/}^2 - 1\ ) / (\ r_{w/}\ r_{0/}'\ )\ \}$

and

$z = \text{Arcsin}\{\ (\ 1 - r_{0/}'\ ) / r_{w/}\ \}$

where $r_{0/}'$ is the starting estimate or 'seed value'. Then:

$\sqrt{[r_{w/}^2 - (r_{0/} - 1)^2]} = r_{0/}\ (\ y + z\ )$

$r_{w/}^2 - (r_{0/} - 1)^2 = r_{0/}^2\ (\ y + z\ )^2$

$(r_{0/} - 1)^2 = r_{w/}^2 - r_{0/}^2\ (\ y + z\ )^2$

$r_{0/}^2 - 2r_{0/} + 1 = r_{w/}^2 - r_{0/}^2\ (\ y + z\ )^2$

Putting the quadratic equation into standard form gives:

$r_{0/}^2\ [1 + (\ y + z\ )^2] - 2r_{0/} + 1 - r_{w/}^2 = 0$

Let:

$t = 1 / [1 + (\ y + z\ )^2]$

Then:

$r_{0/}^2 - 2\ t\ r_{0/} + t\ (1 - r_{w/}^2) = 0$

Thus:

$r_{0/} = \{\ 2t \pm\sqrt{[4t^2 - 4t\ (1 - r_{w/}^2)\ ]}\} / 2$

In this case, the positive sign turns out to be the correct choice, and so:

$r_{0/} = t + \sqrt{[t^2 - t\ (1 - r_{w/}^2)\ ]}$

Iteration is rapidly convergent when the seed value is less than the true value and when overshoot is controlled by feeding back the weighted average:

$r_{0/(new)} = ¼\ r_{0/} + ¾\ r_{0/}'$

The starting approimation used is:

$r_{0/(1st)}' = 1 - r_{w/}^2$

which is that which gives $y = 0$, and will be noted to be the same as equation (**3.1**) (the strained rectangular wire formula for p=0). The round wire calculation produces an effective solenoid radius which is larger than the corresponding rectangular wire value; the reason being that area of an infinitesimal slice parallel to the coil axis diminishes on moving between r_a and the inner solenoid radius (r_a-r_w), and this pinch-off forces r_0 towards r_a .

Note that the routine requires an arcsine function as before. For the convergence criterion shown, the number of iterations reguired is ≤ 24 for $r_w/r_a \leq 0.99$.

```
Function Sscrad(byval rwa as double) as double
'r0/ra for solenoid wound with strained round wire.
'Calling argument is rw/ra.  pitch is neglected.
'v1.00. D W Knight, April 2010.
Dim z as double, y as double, t as double, tt as double, test as double, r0 as double, r0p as double
if rwa >= 1 then
  Sscrad = 0
elseif rwa <= 0 then
  Sscrad = 1
else
  r0p = 1-rwa*rwa
  do
    y = Arcsin((r0p+rwa*rwa-1)/(rwa*r0p))/sqr(1-rwa*rwa)
    z = Arcsin((1-r0p)/rwa)
    t = 1/(1+(y+z)*(y+z))
    tt = t*t-t*(1-rwa*rwa)
    if tt<0 then tt=0
    r0 = t + sqr(tt)
    test=r0-r0p
    r0p=0.25*r0+0.75*r0p
  loop until abs(test) < 1E-9
sscrad= r0p
endif
end function
```

Graphs appear in section **7**, and the calculation is compared with the numerical integration method in the spreadsheet **r0_round_p.ods** (sheet 2).

6. Numerical integration method

By looking back over the previous sections, it can be see that the problem of finding r_0 boils down to that of extracting it from an integral equation of the form:

$$\int_{r_a-r_w}^{r_0} \frac{W(r)\,dr}{(r^2+q^2)^s} = \int_{r_0}^{r_a+r_w} \frac{W(r)\,dr}{(r^2+q^2)^s} \qquad q = p/2\pi \qquad (6.1)$$

where W(r) is a function which describes how the relative width of the conductor varies with r ; and the index s controls the way in which the resistance of an infinitesimal slice of the conductor varies with r. We have so far considered the cases for which s = ½ (homogeneous conductor) and s = 1 (strained conductor); but it should also be obvious that other values of s are possible in the event that (say) the resistivity of the wire is not directly proportional to strain. Also notice that when s = 0, conduction becomes uniform throughout the wire cross section; in which case, if the profile is reflection symmetric about the average solenoid radius r_a , then $r_0 = r_a$. Performing the integration with s=0 is therefore of no practical use for round and rectangular conductors (assuming that a rectangular wire has sides parallel to the axis), but the technique might be used for finding the upper limit of r_0 for a conductor of arbitrary shape.

When the wire is rectangular in cross-section, bearing in mind that constant factors appearing on both sides of the integral equation make no difference to the result, we get:

$$W(r) = 1$$

The integration problem is then a simple one, and analytical solutions for s = ½ and s = 1 have been given as equations (2.1), (2.2), (3.1) and (3.2). When the wire has a circular cross-section however, then:

$$W(r) = \sqrt{[\, r_w^2 - (r-r_a)^2 \,]}$$

In which case, analytical solutions for the integrals at s = ½ or s = 1 exist for p = 0 , but they involve combinations of cyclometric (i.e., inverse trigonometric) functions which do not permit direct separation of r_0. Thus it is necessary to resort to numerical methods even when calculating r_0 in the approximation that $2\pi r_a >> p$.

Inductors used in radio-frequency applications are often wound with thick wire (relative to the coil radius); and for a round-wire solenoid, the pitch to wire diameter ratio (p/d) is always greater than 1 (i.e., $p/r_w > 2$). It can be seen from the rectangular wire calculations given earlier, that the pitch makes a significant contribution to r_0 in tightly wound coils. The same will be true for coils wound with other wire-shapes and so, bearing in mind that inductance is proportional to r_0^2 ; there will be round-wire modelling situations in which it is necessary to take pitch into account.

Given the general difficulty, the most straightforward approach is to perform the integration numerically. To that end, note first of all, that equation (6.1) can be rearranged:

$$\int_{r_a-r_w}^{r_0} \frac{W(r)\,dr}{(r^2+q^2)^s} = 2 \int_{r_a-r_w}^{r_a+r_w} \frac{W(r)\,dr}{(r^2+q^2)^s} \qquad (6.2)$$

The problem then becomes that of first calculating the complete integral on the right and dividing it by 2, then evaluating the integral on the left and adjusting the upper boundary until the result matches the target value. That method is used in the Open Office Basic macro function described below. The process by which the code was developed can be understood by examining the spreadsheets: **r0_rect.ods** (sheet 3) and **r0_numint.ods** (sheets 1-3).

Dimensionless parameters are used for the calculation, as per all of the previous program examples. When all of the parameters are divided by r_a , the complete integral covers the interval between $1-r_{w/}$ and $1+r_{w/}$, where $r_{w/} = r_w/r_a$. Thus the total width of the integration range is $2r_{w/}$, and if we divide this range into k_{max} segments, the width of a segment is given by:

$$\Delta r = 2r_{w/} / k_{max}$$

We now have a set of segments with boundaries lying on the points:

$$r_{k=0} \,,\, r_{k=1} \,,\, . \,.\, . \,.\, ,\, r_{k=kmax}$$

where

$$r_{k+1} - r_k = \Delta r \quad , \quad r_{k=0} = 1-r_{w/} \quad \text{and} \quad r_{kmax} = 1+r_{w/}$$

(Notice here that the ' k= ' part of the subscript has been dropped; except for the k=0 case, where it must be retained in order to avoid confusion with the effective solenoid radius r_0). Now, provided that the segments are 'reasonably narrow' (an issue to be explored later), a good approximation to the integral across any segment is given by Simpson's rule:

$$\int_{k-1}^{k} f_k(r)\, dr \approx \frac{\Delta r}{6} \left[f_{k-1} + 4 f_{k-½} + f_k \right]$$

Where f_{k-1} is the function to be integrated as evaluated at the point $r = r_{k-1}$, etc.; and $f_{k-½}$ is the function as evaluated at the half-way point. Taking the 1/6 : 2/3 : 1/6 weighted average at equally-spaced points covering the integration range is equivalent to fitting the function to a second-order polynomial; and the polynomial is integrated exactly over the segment by multiplying the weighted average by Δr. Hence Simpson's rule gives very accurate results for smooth functions of low order.

Simpson's method is used without modification in the program; but since the calculation involves the comparison of two integrals in order to determine a point on the radius co-ordinate, the factor Δr/6 makes no difference to the outcome and has been dropped.

The complete integral is the sum of the segment integrals:

$$\int_{1-r_{w/}}^{1+r_{w/}} f(r)\, dr = \sum_{k=1}^{k_{max}} \int_{k-1}^{k} f_k(r)\, dr$$

where the radius co-ordinate is taken to be dimensionless (i.e., strictly r/r_a). If we call this integral

$$\Sigma_{kmax}$$

then:

$$\int_{1-r_w}^{r_0} f(r)\, dr = \Sigma_{kmax} / 2$$

The program stores the growing sum Σ_k in an array, each value being obtained by adding the result of a segment integration to the value stored in the preceding element. This enables the segment in which r_0 lies to be determined by stepping through the array until:

$$\Sigma_k \geq \Sigma_{kmax} / 2$$

r_0 is then known to lie between r_{k-1} and r_k (or, extremely rarely, at exactly r_{k-1} ; but the subsequent algorithm works regardless).

A first estimate of r_0 (r_0' say) is obtained by linear interpolation; i.e.;

$$\frac{\int_{k-1,\, r0}}{\int_{k-1,\, k}} = \frac{r_0' - r_{k-1}}{r_k - r_{k-1}} = \frac{r_0' - r_{k-1}}{\Delta r}$$

where $\int_{k-1,\, r0}$ is the integral from r_{k-1} to r_0 , etc. The required quantities are:

$r_{k-1} = 1 - r_{w/} + (k-1)\Delta r$, $\int_{k-1,\, r0} = (\Sigma_{kmax} / 2) - \Sigma_{k-1}$ and $\int_{k-1,\, k} = \Sigma_k - \Sigma_{k-1}$

Hence:

$$r_0' = r_{k-1} + \Delta r \; \{ \; [\; (\Sigma_{kmax}/2 \;) - \Sigma_{k-1} \;] \; / \; (\; \Sigma_k - \Sigma_{k-1} \;) \; \}$$

Linear interpolation is however inaccurate (unless Δr is very small), and so the estimate is refined by a process of iteration. This involves calculating the integral from r_{k-1} to r_0' using Simpson's rule and comparing it with the target value. A new value for r_0' is then obtained by linear interpolation on the difference, and the process is repeated until r_0' and r_0 converge. There is just one subtlety in this matter, without sight of which the algorithm will be difficult to understand; which is that the interpolation integral is carried out on an interval which differs from that of the main integration. Hence, before it can be compared with the results of earlier calculations, it must be scaled by a factor $(r_0' - r_{k-1}) / \Delta r$.

The interpolation procedure converges very rapidly, typically requiring 6 or less cycles to match the integrals to within 12 decimal places. Note however, that the convergence criterion given represents only the accuracy of the interpolation process. The accuracy of the r_0 value returned depends principally on the accuracy of the main integration procedure (from which the target value was obtained).

A prototype version of the program was written to calculate r_0 for coils wound with rectangular wire. The reason for starting in this way was that it allowed verification of the underlying algorithm against the analytical solutions of sections **2** and **3** (see spreadsheet **r0_rect**, sheet 3). A particular motivation in this respect is that the correct coding of the integral denominators for $p > 0$ cannot be verified against analytical solutions in the round-wire case, and so must be tested by some other means.

Once the program was working, it was converted to deal with round wires by the simple expedient of replacing the ones in the integral numerators with a call to the function

Wirew(r_w/r_a , r/r_a , shape)

which calculates the function W(r). Wirew is sufficiently simple that it can be verified by inspection. It is also a useful function in its own right, and was employed in preliminary spreadsheet investigations related to coding of the main program. In this way it was established that the numerical integration method is accurate for round wires (if not yet provably precise), and this knowledge permitted disambiguation of the $p = 0$ analytical solutions of sections **4** and **5**. The iterative routines described in sections **4** and **5** , albeit dependent on the numerical integration for their solution, are inherently precise. Hence (as will be discussed after the program listing) they provide a benchmark for the performance of the numerical integration method.

```
Function swsrad(byval rwa as double, pd as double, s as double, _
sh as integer, kmax as integer) as double
'Effective radius of solid-wire solenoid. v1.01. D W Knight, April 2010.
'rwa=rw/ra.  pd=p/d=p/2rw.  's=0.5 for homogeneous wire, s=1 for strained wire.
'sh=1 for rectangular wire, sh=0 for round wire.
if pd<0 then pd=0
if rwa>1 or (rwa=1 and pd=0) then
  swsrad = 0
elseif rwa <=0 then
  swsrad = 1
else
  if s<0 then s=0
  if kmax<1 then kmax=segcalc(rwa,pd,s,sh)
  if kmax>32766 then kmax=32766
Dim fk(kmax) as double, sum(kmax) as double, qr as double, intgrl as double
Dim rl as double, rm as double, ru as double, fkm as double, n as integer
Dim i as integer, inc as double, target as double, r0 as double
  qr = pd*rwa/pi
'Determine complete integral by dividing into kmax segments and using Simpson's rule.
  inc = 2*rwa/kmax
  sum(0)=0
  ru = 1-rwa
  rm = ru-inc/2
  fk(0) = wirew(rwa,ru,sh)/(ru*ru+qr*qr)^s
  for i = 1 to kmax
    rm = rm+inc
    ru = ru+inc
    fkm = wirew(rwa,rm,sh)/(rm*rm+qr*qr)^s
    fk(i) = wirew(rwa,ru,sh)/(ru*ru+qr*qr)^s
```

```
    sum(i) = sum(i-1) + fk(i-1) + 4*fkm + fk(i)
  next i
'integral from 1-rw/ra to r0/ra is 1/2 the total.
  target = sum(kmax)/2
'find the segment in which r0/ra lies
  for i=1 to kmax
    if sum(i) >= target then exit for
  next i
  rl = 1-rwa+(i-1)*inc
'Use linear interpolation to get first estimate of r0/ra
  r0 = rl + inc*(target-sum(i-1))/(sum(i)-sum(i-1))
'Iterate to refine the estimate.
  for n=1 to 32766
    rm = (r0+rl)/2
    fkm = wirew(rwa,rm,sh)/(rm*rm+qr*qr)^s
    fk(i) = wirew(rwa,r0,sh)/(r0*r0+qr*qr)^s
    intgrl = (fk(i-1) + 4*fkm + fk(i))*(r0-rl)/inc
    r0 = rl + (r0-rl)*(target-sum(i-1))/intgrl
   if abs(sum(i-1)+intgrl-target)<1E-12 then exit for
  next n
  swsrad = r0
endif
end function
```

```
function Wirew(byval rwa as double, rr as double, shape as integer) as double
'Returns relative conductor width. v1.00
'rwa=rw/ra.  rr is relative radius co-ordinate.
'shape=1 for rectangular wire, otherwise assumes round wire.
Dim rrms as double, rwas as double
if shape = 1 then
  wirew=1
else
  rrms=(rr-1)*(rr-1)
  rwas=rwa*rwa
  if rrms>=rwas then
    wirew=0
  else
    wirew=sqr(rwas-rrms)
  endif
endif
end function
```

```
Function segcalc(byval rwa as double, pd as double, s as double, shape as integer) as integer
'Calculates No. of segments for Swsrad to give <100ppM error. v1.00
'shape parameter is not used in this version.
if rwa>=1 and pd<= 0 then
  segcalc = 0
elseif rwa>=1 and pd>0 then
  segcalc = 2048
elseif rwa <= 0.1 then
  segcalc = 10
else
  if s<0 then s=0
  if s>1 then s=1
Dim segs as double
  segs = 2+24.5/(1-rwa^0.25)-863*rwa+5589*rwa^2-18022*rwa^3+30023*rwa^4-
24801*rwa^5+7986*rwa^6
  segs = segs-20
  segs = segs*s
  segs = segs+20
  if segs > 32766 then segs=32766
  if segs < 10 then segs=10
  segcalc = segs
endif
end function
```

For early versions of the program, it was necessary to decide the number of segments to be used in the main integration and pass that information to the program as the parameter k_{max} . That functionality is still available, but if the number of segments requested is zero or less, then the program passes the input parameters to the function 'Segcalc', which attempts to make a reasonable decision. 'Segcalc is written as a separate module so that it can be called independently to find out how many segments are being used.

By comparing the results of the numerical and the analytical calculations, it was established that the worst case error in the numerical integration occurs for the cases where p=0 and r_w/r_a is approaching 1 (this implying a highly distorted current distribution in the wire cross section). It was also confirmed that the number of segments for a given accuracy is is always greater for the strained round wire case (s=1) than it is for the homogeneous case (s=0.5). Thus the p = 0 , s=1 worst case was chosen as the basis on which to calculate the number of segments required.

For parameters sets relating to realistic coils, the required number of segments is remarkably small; being generally less than 20 for better than 100ppM (0.01%) accuracy. This means that the program can be made fast to respond in practical modelling situations. As $r_w/r_a \rightarrow 1$ with p = 0 however, the number of segments needed increases dramatically; which means that the program will become unacceptably inaccurate in extreme cases if a low fixed number of segments is used.

The formula used in the function 'Segcalc' was obtained as follows: The error in the p = 0 , s = 1 case was obtained by comparing the result with that calculated using the function 'Sscrad' (section **5**) on 50 points over the r_w/r_a range from about 0.955 to 0.1 (see spreadsheet **r0_numint**, sheet 2). The number of segments used for each calculation was then adjusted by hand until an error of just less than 100ppM was obtained. This set of k_{max} values for < 100ppM error was then fitted to the function:

$F(r_{w/}) = F_0/(1-r_{w/}^{0.25}) + F_1\, r_{w/} + F_2\, r_{w/}^2 + F_3\, r_{w/}^3 + F_4\, r_{w/}^4 + F_5\, r_{w/}^5 + F_6\, r_{w/}^6$

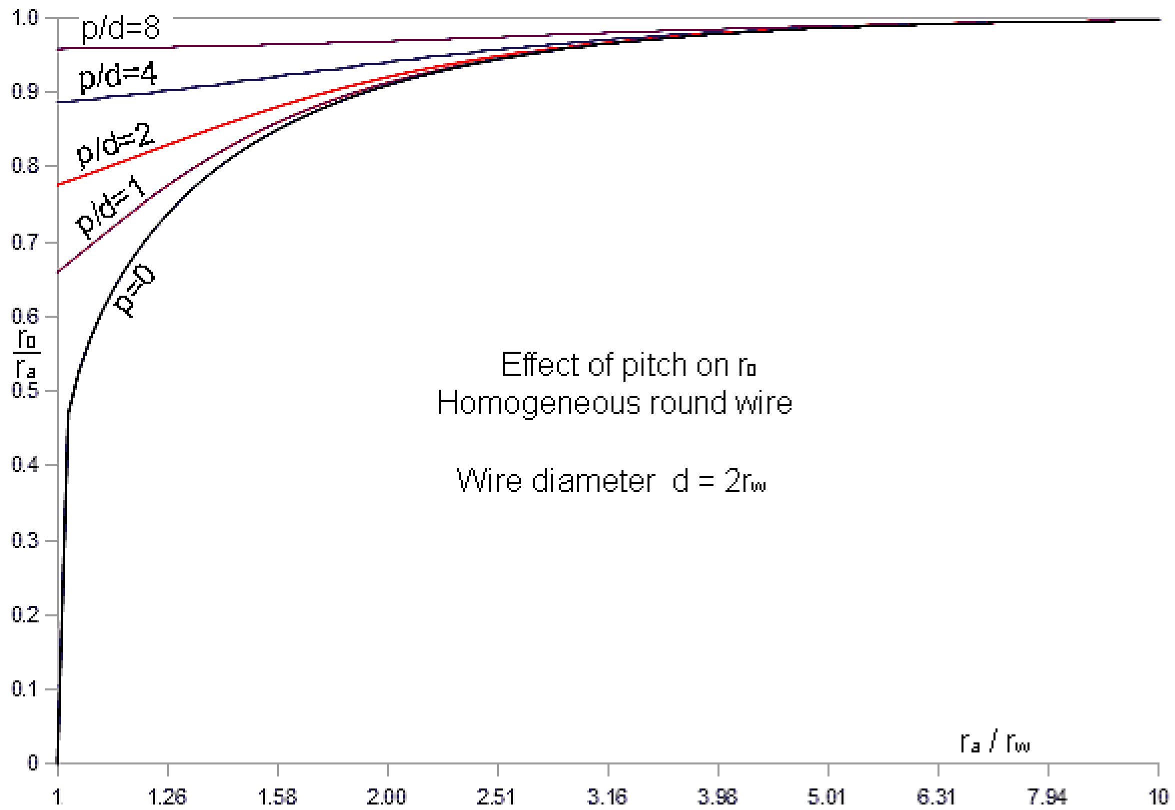
The fitting procedure gave a maximum residual of < 2, and so, the number of segments needed to guarantee < 100ppM error is the integer converted value of $F(r_{w/})$+2.

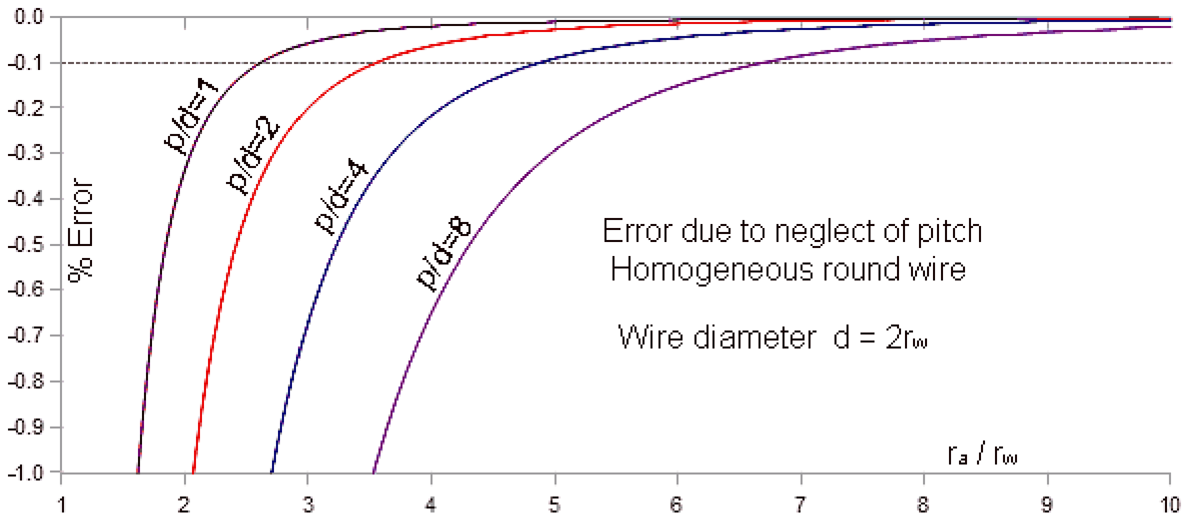
The segment allocation for the s=1 case is, of course, overgenerous for the s = 0.5 case. To reduce the response time for s < 1 , the program subtracts 20 from the result for the s = 1 case, multiplies the remainder by s , then adds 20. This is not an accurate method for determining just enough segments for < 100ppM error (it is still overgenerous), but it is an optimisation in aid of calculations involving multiple function calls (such as graph plotting) to be had for minimal developmental effort.

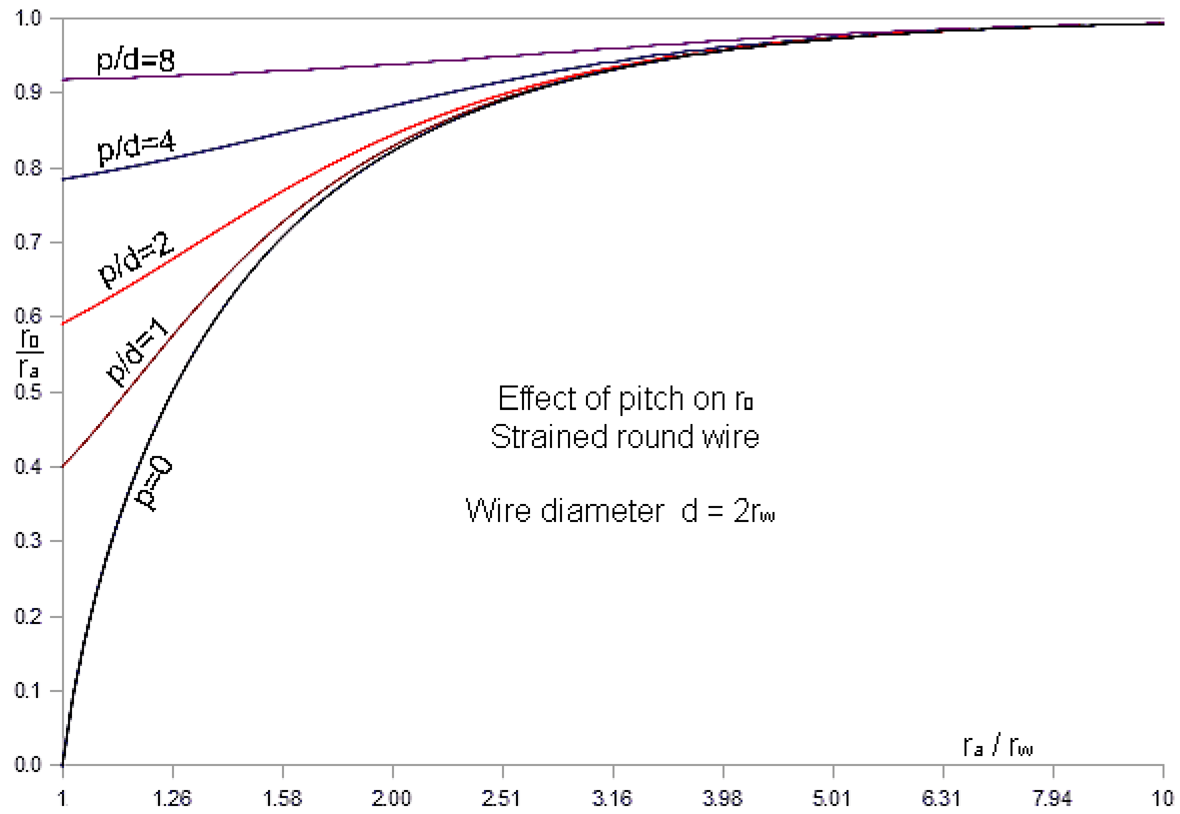
Note that the number of segments used in the numerical integration is never allowed to fall below 10. This was taken to be the limit below which it is not worth bothering to try to optimise the response time; although, for realistic rectangular wire calculations, accurate results can be obtained using only one segment.

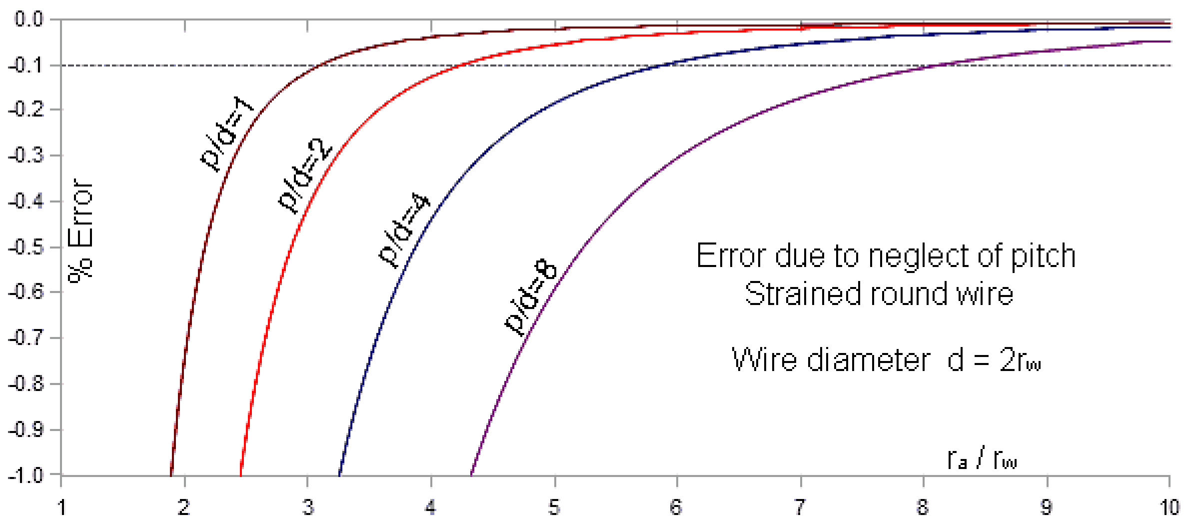
7. Effect of pitch on r_0 of round-wire solenoids.

The graphs shown below are taken from the spreadsheet **r0_round_p.ods** (sheet 1 for the homogeneous wire case and sheet 2 for the strained wire case). They use the numerical integrator "Swsrad" to determine the effect of pitch (and the effect of neglecting pitch) in the calculation of r_0 for round wire solenoids. What cannot be seen by looking at the graphs is that the p=0 cases are actually two curves superimposed, one being produced by corresponding semi-analytical routine of section **4** or **5**. The numerical method is accurate within 0.01% when the automatic segmentation option is used.









Note that for these graphs, the winding pitch is in units of wire diameter. It is not physically possible to make round wire coils with p/d < 1; and in practical coils, p/d is always > 1 due to the need for wire insulation and because the pitch distance is strictly the conductor centre-to centre distance divided by the cosine of the pitch angle[8].

8 See: **Numerical Methods for Inductance Calculation**, Bob Weaver, http://electronbunker.ca/CalcMethods2b.html

8. Further possibilities

The work discussed here stops at a point which satisfies the author's inductor modelling requirements. The numerical integrator routine however, can easily be adapted to deal with a range of other wire shapes by modifying the function "Wirew" and adding more allowed values to the calling parameter 'sh' or 'shape'. As it stands, the program can give a correct result for any W(r) function which is differentiable on the interval from $1-r_{w/}$ to $1+ r_{w/}$. Thus the method might be used (say) to model coils with triangular or trapezoidal wire, provided that the shape is arranged to satisfy the requirement of differentiability (i.e., the triangle has one flat side parallel to the coil axis, etc.). Elliptical wire can also be modelled, but that will require an additional shape parameter.

In the case where the W(r) function is not differentiable over the required interval, the method cannot be used as it stands. There may be reason to extend it however, in order to deal with tubes and plated wires. In that case, the main integration process must be split into zones, each having a differentiable W(r) function associated with it. Thus, for example, a plated wire or a tube, will require three zones because there are three distinct regions on moving between $1-r_{w/}$ and $1+ r_{w/}$. The trick in getting this to work lies in segmenting the zones separately, so that each has a segment boundary at the upper and lower limit. Note then the need to scale the separate integrals according to segment width before they can be added together.

An interesting side-effect of dividing the main integral into zones is that it allows different degrees of strain in the various layers of the conductor. Thus it becomes possible to model the difference between a coil wound with plated wire (strained coating) and a coil of wire which is plated after the helix is formed (homogeneous coating). It will be understood, of course, that winding tight coils with pre-plated wire is generally a bad idea; and in some cases the coating can be seen to have ruptured entirely.

99. Discussion

Fraga et al. [cited earlier] derive the effective radius at low frequencies for a coil wound with homogeneous rectangular wire and use it as a model for a round wire solenoid. It causes a noticeable improvement in the accuracy of their predictions. Hence the common assumption that the effective radius is the same as the average radius is a cause of systematic error in inductance calculations; albeit a minor one when the radius of the coil is considerably greater than the radius of the wire.

Here we take the issue a little further and argue that straining the wire to wrap it around a coil former sets up a resistivity gradient, reducing the effective radius again. Also we note that the use of the analytically-tractable rectangular wire model is not particularly accurate for tight round-wire coils, and the effect of pitch can become significant. A numerical integration method is given as an alternative, and only becomes slow to calculate in extreme cases of little practical importance.

* * *

The numerical routines discussed in the text can be copied from the accompanying Open Document spreadsheet file: **r0_numint.ods**. To access the code (assuming the use of Open Office 3 software), use the ' Tools > Macros > Organise Macros > OOo Basic ' menu.